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Small inertial effects on a spherical bubble, drop or particle moving near a wall in a time-dependent linear flow

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The problem of a spherical drop of arbitrary density and viscosity moving near a wall under the effect of a body force is analysed theoretically in the limit where the wall lies in the inner region of the flow disturbance, the distance between the drop and the wall being large compared to the drop radius. The drop may move in an arbitrary direction with respect to the wall, and the undisturbed flow field is assumed to comprise a steady uniform shear or solid-body rotation and a time-dependent uniform stream, the variations of which take place over time scales large compared to the viscous diffusion time. An exact force balance with no limitation on the magnitude of inertial effects is obtained by using the reciprocal theorem. Explicit expressions for the contributions of temporal acceleration, slip and shear or rotation to the total hydrodynamic force are derived in the limit of small-but-finite inertial effects. The connection between these near-wall results and inertial lift and drag corrections in an unbounded flow is discussed. Situations of particular interest in which the lift force results from a combination of contributions due to unsteadiness and advection, like the case of a particle moving near the bottom wall of a centrifuge, are also examined.

1. Introduction

The determination of inertial forces acting on small rigid particles moving near a wall has been a classical problem of low-Reynolds-number hydrodynamics for about forty years. The initial impulse in this field was given by the pioneering experiments of Segré & Silberberg (1962*a*, *b*) who demonstrated the existence of a lateral migration of small neutrally buoyant spheres transported by a Poiseuille flow. After Bretherton (1962) recognized that inertial effects must be taken into account to explain the existence of a sideways force on a sphere, two main streams of research developed. One of them focused on inertial interactions between the flow disturbance produced by the particle and the wall, while the other neglected the direct influence of the wall and concentrated on inertial effects in the far field of the disturbance. The former approach is relevant to situations in which the wall lies in the inner (Stokes) region of the disturbance, so that the dominant inertial contribution results from a regular perturbation of the creeping flow solution. The second approach finds applications in cases where Oseen-like corrections dominate over wall-induced effects; in this case the leading-order inertial effects result from the outer expansion of the disturbance and their evaluation requires the solution of a singular perturbation problem.

Two early studies typical of the first series of work are those of Ho & Leal (1974) and Vasseur & Cox (1976) who considered the case of plane Couette and Poiseuille flows. The prototype of the second stream of studies is that of Saffman (1965) in which the expression for the lift force acting on a small sphere moving along the streamlines of an unbounded linear shear flow was derived under conditions where inertial effects due to the shear dominate those due to the slip velocity. An intermediate regime in which the wall lies in the outer region of the flow disturbance but has still a noticeable influence on the lateral migration of the particle was considered by Vasseur & Cox (1977) for the case of a fluid at rest at infinity, by McLaughlin (1993) for a linear shear flow, and by Schonberg & Hinch (1989) and Hogg (1994) for plane Poiseuille flow. Theoretical and experimental studies concerned with wall-induced migration have been reviewed by Leal (1980) and Hogg (1994). Contributions arising from Saffman's (1965) paper and devoted to lateral migration and Oseen-like drag corrections in unbounded linear flows have recently been reviewed by Stone (2000).

All the results derived in the aforementioned studies concern rigid particles and it is only recently that inertial corrections affecting spherical or spheroidal drops and bubbles have been considered. The reason seems to be that many experiments involving droplets and small bubbles, especially those focused on suspension rheology, have been performed in liquids of high viscosity where the lateral migration due to deformation dominates that due to inertia. Nevertheless it is easy to show that in many practical situations (say e.g. those in which the viscosity of the suspending liquid is up to one hundred times that of water for millimetric drops), inertial effects are dominant. Consequently, determining wall-induced and Oseen-like or Saffman-like inertial effects due to temporal acceleration and advection for drops and bubbles appears to be of importance for a wide range of applications. Among them one can mention deposition of droplets on walls, separation techniques such as centrifugation, field-flow fractionation (FFF) or sedimentation fractionation, nucleate boiling, etc. After a comprehensive investigation of the effects of temporal acceleration on a rigid sphere moving in a uniform unbounded flow in the Oseen regime (Lovalenti & Brady 1993*a*), Lovalenti & Brady (1993*b*) performed a similar analysis for the case of a drop of arbitrary viscosity. Similarly, Legendre & Magnaudet (1997) extended Saffman's (1965) result to the case of a spherical drop or bubble. These two contributions clearly belong to the second series of work mentioned above since they did not consider any wall influence. In contrast, in a recent investigation, Magnaudet, Takagi & Legendre (2003, hereinafter referred to as MTL), examined the inertial (and deformation-induced) migration of a buoyant drop of arbitrary viscosity moving in a quiescent fluid or in a linear shear flow bounded by a single wall located in the inner region of the flow disturbance. Their results concerning the inertial migration extend those of Cox & Hsu (1977) and Cherukat & McLaughlin (1994) to drops and bubbles; they are also slightly more general in that they consider the case of a vertical or horizontal wall, i.e. the leading-order slip velocity of the drop is allowed to be parallel or perpendicular to the wall.

The theoretical framework of the present investigation is similar to that of §§ 6 and 7 of MTL, i.e. we consider small inertial effects acting on a spherical drop of arbitrary viscosity moving near a flat wall located in the inner region of the disturbance. Nevertheless the present work broadens significantly that of MTL by considering a wider class of flows. Our main goal is to establish a general force balance on a drop in arbitrary motion in a linear flow bounded by a single wall and to obtain an explicit expression for the inertial forces due to temporal acceleration, slip, uniform shear or solid-body rotation, these two particular families of linear flows being the only ones compatible with the no-slip condition on a rigid wall. The organization of the paper is as follows. The governing equations of the problem and the main assumptions are established and discussed in § 2. In § 3 we analyse

	Inviscid bubble ($\overline{\rho}=0, \lambda=0$)	Rigid sphere ($\lambda \to \infty$)
Quasi-steady Stokes drag, $\boldsymbol{F}_{DS}$	$-4\pi\left\{\left(1+\frac{3}{8}\kappa+\frac{9}{64}\kappa^2+\frac{27}{512}\kappa^3\right)\boldsymbol{V}_{S0}^{\parallel} +\left(1+\frac{3}{4}\kappa+\frac{9}{16}\kappa^2+\frac{27}{64}\kappa^3\right)\boldsymbol{V}_{S0}^{\perp}\right\}$	$-6\pi\left\{\left(1+\frac{9}{16}\kappa+\frac{81}{256}\kappa^2+\frac{217}{4096}\kappa^3\right)\boldsymbol{V}_{S0}^{\parallel} +\left(1+\frac{9}{8}\kappa+\frac{81}{64}\kappa^2+\frac{473}{512}\kappa^3\right)\boldsymbol{V}_{S0}^{\perp}\right\}$
Shear-induced Faxén force, $\boldsymbol{F}_F$	$-\frac{\pi}{2}\alpha\left(1+\frac{3}{8}\kappa\right)\kappa^2\boldsymbol{e}_1$	$-\frac{15}{8}\pi\alpha\left(1+\frac{9}{16}\kappa\right)\kappa^2\boldsymbol{e}_1$
Long-time reaction to temporal acceleration, $\boldsymbol{F}_U$	$-\pi Re\,St\left\{\left[3\kappa^{-1}+\frac{1}{4}\right]\frac{\mathrm{d}\boldsymbol{V}_{S0}^{\parallel}}{\mathrm{d}t} +\left[\kappa^{-1}-\frac{1}{2}\right]\frac{\mathrm{d}\boldsymbol{V}_{S0}^{\perp}}{\mathrm{d}t}\right\}$	$-\frac{9}{4}\pi Re\,St\left\{\left[3\kappa^{-1}+\frac{217}{216}\right]\frac{\mathrm{d}\boldsymbol{V}_{S0}^{\parallel}}{\mathrm{d}t} +\left[\kappa^{-1}-\frac{13}{108}\right]\frac{\mathrm{d}\boldsymbol{V}_{S0}^{\perp}}{\mathrm{d}t}\right\}$
Slip-induced lift $(\boldsymbol{F}_L)_{Sl}$	$\frac{\pi}{4}Re(\boldsymbol{V}_{S0}^{\parallel}\cdot\boldsymbol{V}_{S0}^{\parallel})\boldsymbol{e}_3$	$\frac{9}{16}\pi Re(\boldsymbol{V}_{S0}^{\parallel}\cdot\boldsymbol{V}_{S0}^{\parallel})\boldsymbol{e}_3$
Shear-induced lift, $(\boldsymbol{F}_L)_{Sh}$	$-\frac{\pi}{8}\alpha Re\left\{5\left(\kappa^{-1}-\frac{121}{120}\right)(\boldsymbol{V}_{S0}\cdot\boldsymbol{e}_3)\boldsymbol{e}_1 +\frac{11}{3}\left[\left(\kappa^{-1}+\frac{99}{88}\right)(\boldsymbol{V}_{S0}\cdot\boldsymbol{e}_1)-\frac{\alpha}{3}\right]\boldsymbol{e}_3\right\}$	$-\frac{9}{32}\pi\alpha Re\left\{5\left(\kappa^{-1}-\frac{2191}{10800}\right)(\boldsymbol{V}_{S0}\cdot\boldsymbol{e}_3)\boldsymbol{e}_1 +\frac{11}{3}\left[\left(\kappa^{-1}+\frac{2215}{2640}\right)(\boldsymbol{V}_{S0}\cdot\boldsymbol{e}_1)-\frac{5}{9}\alpha\right]\boldsymbol{e}_3\right\}$
Rotation-induced lift, $(\boldsymbol{F}_L)_R$	$\frac{\pi}{4}Ta\left[\frac{27}{16}\left(\kappa^{-1}+\frac{3}{4}\right)+\frac{16}{3}\right]\boldsymbol{e}_3\times\boldsymbol{V}_{S0}$	$\frac{\pi}{4}Ta\left[\frac{243}{64}\left(\kappa^{-1}+\frac{9}{8}\right)+\frac{16}{3}\right]\boldsymbol{e}_3\times\boldsymbol{V}_{S0}$

TABLE 1. Summary of zero-Reynolds-number and leading-order inertial contributions to the force experienced by an inviscid massless bubble or a rigid sphere. The quasi-steady Stokes drag is obtained by using (A 3*a*) and (A 6) (see also MTL for higher-order approximations); the shear-induced Faxén force is given in (12); the expression for the long-time reaction to temporal acceleration given in (14*a*, *b*) is valid provided $Re\,St \ll 1$; the three contributions to the quasi-steady inertial lift force derived under conditions $Re \ll 1, \alpha Re \ll 1$ and $Ta \ll 1$ are given in (17) and (21); effects of temporal acceleration and quasi-steady inertial effects may be added provided $Re \ll St \ll Re^{-1/2}$ (see (6)).

the low-Reynolds-number perturbation problem and show why, within the present assumptions, the outer expansion does not contribute to the leading-order inertial corrections to the hydrodynamic force. Section 4 describes the so-called auxiliary problem and shows how the general force balance on the drop (not limited to low Reynolds numbers) may be obtained by applying the reciprocal theorem. Section 5 describes the results concerning inertial forces induced by temporal acceleration, while § 6 (resp. § 7) analyses those due to slip and shear (resp. slip and solid-body rotation). In these three sections we discuss the connection between the present results valid near a wall and their counterpart in an unbounded flow. We also show how the effects of temporal acceleration and quasi-steady advection combine in some situations of particular interest, like that of a drop moving in a centrifuge. Some concluding remarks are given in § 8 and the main results of the present investigation are summarized in table 1 for the two limit cases of a rigid sphere and a bubble of negligible density and viscosity. Since the material required to obtain the solution of the auxiliary problem and to evaluate the particle-induced disturbance near a wall was established in MTL, technical details are not repeated here. Nevertheless an outline of the corresponding results is given in Appendices A and C. Some details concerning the derivation of the reciprocal theorem are given in Appendix B.

2. Assumptions and governing equations

Let us consider a drop of radius R made of a Newtonian fluid of density $\tilde{\rho}$ and viscosity $\tilde{\mu}$ moving in a suspending Newtonian fluid of density ρ and viscosity μ

bounded by an infinite wall. Throughout this work, the density ratio $\bar{\rho} = \tilde{\rho}/\rho$ and the viscosity ratio $\lambda = \tilde{\mu}/\mu$ are arbitrary. We assume that the time-dependent distance L separating the drop centre from the wall is much larger than R, so that the length ratio $\kappa = R/L$ is small compared to unity. We normalize distances by R and velocities by a velocity scale V_C characterizing the slip of the drop with respect to the local undisturbed flow. In addition to κ, the problem involves several other control parameters, in particular the Reynolds number $Re = \rho V_C R/\mu$ and the product $Re\,St$ where the generalised Strouhal number St is defined as $St = R/\tau V_C$, τ being the time scale characterizing possible effects of temporal acceleration. We assume that Re, $Re\,St$, $(\bar{\rho}/\lambda)Re$ and $(\bar{\rho}/\lambda)Re\,St$ are small compared to unity, so that at leading order the problem under consideration is governed by the steady Stokes equations. Clearly, capillary effects resulting from a finite surface tension γ are required to satisfy the normal-stress balance at the drop surface; these effects may be characterized by a capillary number $Ca = \mu V_C/\gamma$. Non-zero values of Ca induce a deformation of the drop from which $O(Ca)$-corrections to the total hydrodynamic force may result. Deformation-induced forces experienced by a drop moving at zero Reynolds number near a wall have been extensively studied in the past (see e.g. Chan & Leal 1979; MTL and references therein). Here we focus on inertial effects, so that the condition $Ca \ll \min(Re, Re\,St)$ is assumed to be satisfied throughout this work. Under this condition it may be shown that the inertial effects to be considered in this work are larger than those due to deformation, so that it is legitimate to consider that the drop maintains a spherical shape; in MTL this situation was shown to be very common for moderately viscous suspending fluids. The above assumptions may be summarized in terms of a hierarchy of time scales, namely we assume that the capillary time $\mu R/\gamma$ is much smaller than the viscous time $\rho R^2/\mu$, which itself is much smaller than both the advective time R/V_C and the time τ of temporal acceleration.

The restriction $\rho R^2/\mu \ll \tau$ (equivalent to $Re\,St \ll 1$) implies that the results to be derived below may not apply to the initial (resp. final) $O(\rho R^2/\mu)$-stage of the start (resp. stop) of the drop motion; this is especially true if $\tau \to 0$, as in the case of a sudden start or stop. Similarly, in the case where the drop undergoes a purely periodic motion, these results cannot be used to describe the $O(\rho R^2/\mu)$-part of each period surrounding the zero-crossings of the slip velocity because the temporal acceleration is not small compared to the viscous term during this part of the motion. These restrictions arise because we do not solve the unsteady Stokes equations, which would be required to obtain uniformly valid predictions. Solving these equations for a drop near a wall appears to be a considerable task. To the best of our knowledge the only attempt in this direction was performed by Wakiya (see Happel & Brenner 1973, p. 354), who obtained the leading-order wall correction to the Basset–Boussinesq equation for a rigid sphere translating parallel to a wall in a quiescent fluid. Moreover, in an unbounded flow the solution of the unsteady Stokes equations may only be put into a closed form as a function of time in the two limits $\lambda = 0$ and $\lambda \to \infty$ (Yang & Leal 1991; Lovalenti & Brady 1993*b*). Consequently, in presence of a wall there is little hope of obtaining formulae of practical use in the case of rapidly varying flow conditions and we shall restrict ourselves to the assumptions stated above.

In MTL, as in the present work, the influence of the wall is taken into account by using the method of reflections (Happel & Brenner 1973, Chaps. 6 and 7). In this technique, the length ratio κ is assumed to be small and the minimum separation distance from the wall, $\kappa_{\min}^{-1}(n)$, at which results obtained by truncating the solution arbitrarily at $O(\kappa^n)$ are valid is not known theoretically. Based on comparisons with exact solutions or experiments, MTL noticed that the $O(\kappa^3)$-truncation of the

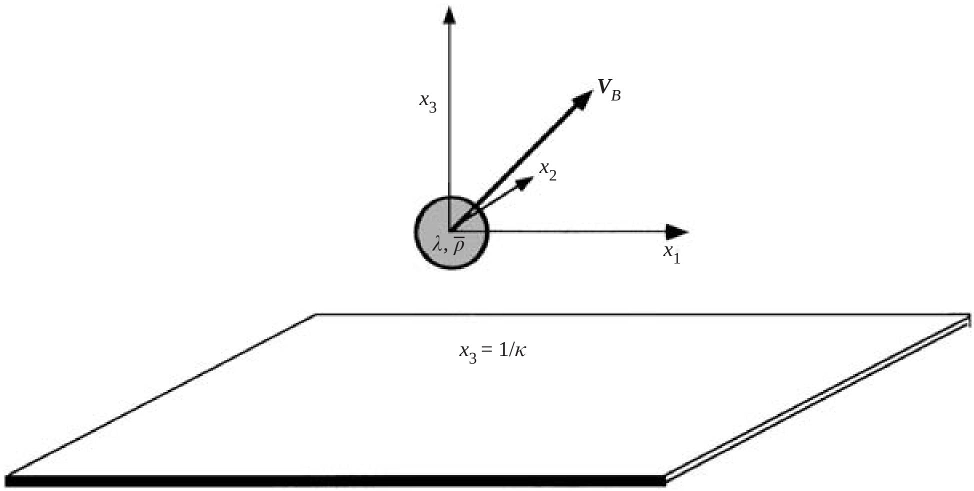

FIGURE 1. Sketch of the problem and coordinate system.

creeping-flow solution predicts the drag force on the drop with an accuracy of a few percent provided κ is less than 0.5, i.e. the thickness of the 'film' between the drop and the wall is larger than the drop radius. Moreover, they observed that in the limit $\kappa = 1$ and $\lambda \to \infty$, corresponding to the situation of a rigid particle touching the wall, the $O(\kappa^0)$-approximation of the slip-induced and shear-induced lift force agrees within 5% with the exact solution derived by Krishnan & Leighton (1995). Based on these remarks, we shall truncate the base creeping-flow solution at $O(\kappa^3)$ and the inertial corrections at $O(\kappa^0)$ throughout the present investigation.

To formulate the problem we use a Cartesian coordinate system $(Ox_1x_2x_3)$ centred at the instantaneous position of the centre O of the drop and translating with it. The unit vectors corresponding to directions x_1, x_2 and x_3 are $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, respectively, with x_3 perpendicular to the wall and directed away from it (figure 1). The local distance to O is $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and the radial unit vector is $\boldsymbol{e}_r = \boldsymbol{x}/r$ with $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$. Note that with this choice of coordinates the wall is located at $x_3 = -1/\kappa$.

Let $\boldsymbol{U}$ (resp. $\tilde{\boldsymbol{U}}$) be the *relative* velocity of the outer (resp. inner) fluid with respect to O and $\boldsymbol{V}_B$ (resp. $\boldsymbol{V}_W$) be the *absolute* velocity of the drop (resp. wall). The undisturbed flow is characterized by an absolute velocity field $\boldsymbol{V}(\boldsymbol{x}, t)$ to be specified later. For $x_3 = -1/\kappa$, the kinematic and no-slip boundary conditions imply $\boldsymbol{V} = \boldsymbol{V}_W$. We then write the governing equations in the form

$$\left.\begin{array}{l}
\nabla\cdot\boldsymbol{U} = 0 \quad \nabla\cdot\tilde{\boldsymbol{U}} = 0,\\
\nabla\cdot\boldsymbol{\Sigma} = Re\left(St\dfrac{\partial\boldsymbol{U}}{\partial t} + \boldsymbol{U}\cdot\nabla\boldsymbol{U}\right), \quad \nabla\cdot\tilde{\boldsymbol{\Sigma}} = \dfrac{\bar{\rho}Re}{\lambda}\left(St\dfrac{\partial\tilde{\boldsymbol{U}}}{\partial t} + \tilde{\boldsymbol{U}}\cdot\nabla\tilde{\boldsymbol{U}}\right),\\
\boldsymbol{U} = \boldsymbol{V}_W - \boldsymbol{V}_B \quad \text{for} \quad x_3 = -1/\kappa,\\
\boldsymbol{U} \to \boldsymbol{V} - \boldsymbol{V}_B \quad \text{for} \quad r \to \infty,\\
\left.\begin{array}{l}
\boldsymbol{U}\cdot\boldsymbol{n} = \tilde{\boldsymbol{U}}\cdot\boldsymbol{n} = 0\\
\boldsymbol{n}\times\boldsymbol{U} = \boldsymbol{n}\times\tilde{\boldsymbol{U}}\\
\boldsymbol{\Sigma}\cdot\boldsymbol{n} = \lambda\tilde{\boldsymbol{\Sigma}}\cdot\boldsymbol{n} + (1-\bar{\rho})\Phi\boldsymbol{n} + \dfrac{1}{Ca}(\nabla\cdot\boldsymbol{n})\boldsymbol{n}
\end{array}\right\} \quad \text{for} \quad r = 1 + O(Ca),
\end{array}\right\} \qquad (1)$$

where $\boldsymbol{\Sigma} = -P\boldsymbol{I} + \nabla \boldsymbol{U} + \nabla^T \boldsymbol{U}$ (resp. $\tilde{\boldsymbol{\Sigma}} = -\tilde{P}\boldsymbol{I} + \nabla \tilde{\boldsymbol{U}} + \nabla^T \tilde{\boldsymbol{U}}$) is the stress tensor (normalized by $\mu V_C/R$) in the outer (resp. inner) fluid and $\boldsymbol{n}$ is the unit normal to the drop surface directed into the suspending fluid, $\boldsymbol{I}$ denoting the Kronecker tensor and ∇^T the transpose of the ∇ operator. The condition at $x_3 = -1/\kappa$ is just the no-slip condition at the wall, while that for $r \to \infty$ indicates that the disturbance induced by the drop must vanish at large distances. The first boundary condition at the drop surface expresses the fact that the normal velocity is zero because the drop does not deform. The other two conditions result from the matching of tangential velocities and stresses across the drop surface, respectively. In the above formulation, the pressure P (resp. $\tilde{P}$) involves a potential Φ (resp. $\bar{\rho}\Phi$) defined as $\nabla\Phi = \boldsymbol{F} = \boldsymbol{g} - Re\, St\, \mathrm{d}\boldsymbol{V}_B/\mathrm{d}t$ where $\boldsymbol{g}$ is the dimensionless body force (such as gravity) and $Re\, St\, \mathrm{d}\boldsymbol{V}_B/\mathrm{d}t$ is the complementary acceleration, required because the reference frame attached to the drop is generally non-inertial. Integrating the momentum equation within the drop, noting that the volume integral of $\partial\tilde{\boldsymbol{U}}/\partial t$ is zero because the volume-averaged velocity of the fluid inside the drop is $\boldsymbol{V}_B$ for all t, and using the last of (1) with the constraint of constant surface tension, we obtain the force balance on the drop as

$$\frac{4}{3}\pi(\bar{\rho} - 1)\boldsymbol{F} + \int_{A_B} \boldsymbol{\Sigma}\cdot\boldsymbol{n}\,\mathrm{d}S = \boldsymbol{0}, \tag{2}$$

where A_B denotes the drop surface.

To complete the specification of the problem we must prescribe the form of the undisturbed flow $\boldsymbol{V}$ and that of the wall velocity $\boldsymbol{V}_W$ that ensues. We shall first consider the case of a linear shear flow with a dimensional shear rate $\alpha V_C/R$. Then we have

$$\boldsymbol{V}(\boldsymbol{x}, t) = \boldsymbol{V}_\mathrm{W}(t) + \alpha\left(x_3 + \frac{1}{\kappa(t)}\right)\boldsymbol{e}_1, \tag{3}$$

where the notation $\kappa(t)$ indicates that the distance between the drop and the wall may vary in time. Then we shall allow $\boldsymbol{V}_W$ to be the sum of a translation $\boldsymbol{V}_\Omega(t)$ and a solid-body rotation with a dimensional rotation rate $\Omega V_C/R$ about an axis parallel to x_3 that crosses the wall at $\boldsymbol{x}_\Omega = -(x_{10}(t), x_{20}(t), 1/\kappa(t))$. In this case we have

$$\boldsymbol{V}(\boldsymbol{x}, t) = \boldsymbol{V}_W(\boldsymbol{x}, t) = \boldsymbol{V}_\Omega(t) + \Omega((x_1 + x_{10}(t))\boldsymbol{e}_2 - (x_2 + x_{20}(t))\boldsymbol{e}_1). \tag{4}$$

Note that the velocity fields defined by (3) and (4) cannot be added because the resulting flow would not be a solution of the Navier–Stokes equations since the flow acceleration would not be irrotational. Thus in the rest of this paper the flow may comprise homogeneous time-dependent contributions in each direction, a steady linear shear in a direction parallel to the wall or a steady rotation about the direction normal to it.

3. The low-Reynolds-number perturbation problem

Let us first re-arrange the first of the momentum equations in (1) by splitting $\boldsymbol{U}$ in the form $\boldsymbol{U} = \boldsymbol{V} - \boldsymbol{V}_B + \boldsymbol{u}$, where $\boldsymbol{u}$ is the velocity disturbance in the suspending fluid. This allows us to write

$$St\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = \underbrace{St\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{V}}_{(\mathrm{I})} + \underbrace{St\frac{\partial \boldsymbol{V}}{\partial t} + ((\boldsymbol{V} - \boldsymbol{V}_B)\cdot\nabla)\boldsymbol{V}}_{(\mathrm{II})} - St\frac{\mathrm{d}\boldsymbol{V}_B}{\mathrm{d}t}. \tag{5}$$

The group of terms (I) is the part of the fluid acceleration involving the flow disturbance; it will be frequently denoted by $\boldsymbol{f}$ for compactness. Since the connection between the relative position $\boldsymbol{x}$ and the absolute position $\boldsymbol{x}'$ is

$$\boldsymbol{x}' = \boldsymbol{x} + \int_0^{t'} \boldsymbol{V}_B(u)\,\mathrm{d}u,$$

it is easy to see that the group of terms (II) simply the fluid acceleration

$$\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t'} = St\frac{\partial \boldsymbol{V}}{\partial t'} + \left(\boldsymbol{V}\cdot\nabla'\right)\boldsymbol{V},$$

the primes denoting spatial coordinates and time evaluated in the absolute frame of reference. Coming back to (1) we see that it is also convenient to split the stress tensor $\boldsymbol{\Sigma}$ in the form $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 + \boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ is the stress tensor associated to the flow disturbance (thus satisfying $\nabla\cdot\boldsymbol{\sigma} = Re\boldsymbol{f}$). Then we may write $\boldsymbol{\Sigma}_0 = \Psi\boldsymbol{I} + 2\boldsymbol{S}$ where the potential Ψ is such that $\nabla\Psi = Re(\mathrm{D}\boldsymbol{V}/\mathrm{D}t' - St\,\mathrm{d}\boldsymbol{V}_B/\mathrm{d}t)$ and $\boldsymbol{S}$ is the strain-rate tensor of the undisturbed flow, i.e. $\boldsymbol{S} = \frac{1}{2}(\nabla\boldsymbol{V} + {}^T\nabla\boldsymbol{V})$.

We now come to the specific situation we have in mind and assume that Re and $Re\,St$ are small compared to unity. Examining the disturbance momentum equation $\nabla\cdot\boldsymbol{\sigma} = Re\boldsymbol{f}$ we see that unsteady effects and slip-induced inertial effects become comparable to viscous effects at distances $l_u = O((Re\,St)^{-1/2})$ and $l_s = O(Re^{-1})$, respectively. Similarly, effects of shear or rotation become comparable to viscous effects at $l_\alpha = O((\alpha Re)^{-1/2})$ and $l_\Omega = O((\Omega Re)^{-1/2})$, respectively. Then, if the separation distance κ^{-1} between the drop and the wall is such that $\kappa^{-1} < \min(l_u, l_s, l_\alpha, l_\Omega)$, the flow is correctly described by the steady Stokes equations in the wall region. Cox & Brenner (1968) recognized that in this situation Stokes solutions decay like r^{-2} for $r \gg \kappa^{-1}$ instead of decaying like r^{-1} in an unbounded flow; this behaviour, which holds whatever the orientation of the particle motion with respect to the wall, is due to the effect of the image velocity field produced by the wall. Despite this faster decay, the region $r > \min(l_u, l_s, l_\alpha, l_\Omega)$ is an outer region and the overall perturbation associated with inertial effects is singular. Nevertheless Cox & Hsu (1977) showed that in a large class of flows, the outer region does not contribute to first-order inertial corrections. Their argument can easily be extended to the unsteady situation considered here.

For this, let us consider again the disturbance momentum equation $\nabla\cdot\boldsymbol{\sigma} = Re\boldsymbol{f}$. At leading order in Re and $Re\,St$, its solution in the inner region is the sum of a particular solution corresponding to the forcing by the term $\boldsymbol{f}$ evaluated from the creeping-flow solution, and of a complementary solution satisfying the homogeneous equation $\nabla\cdot\boldsymbol{\sigma}_{CS} = \boldsymbol{0}$, both solutions having to satisfy separately the vanishing of the velocity disturbance at the wall. Since the leading-order terms of $\boldsymbol{f}$ are of $O(Re\,r^{-2})$ and $O(Re\,St\,r^{-2})$ for $r \to \infty$, it follows that, just like in the classical Oseen problem, the leading terms of the particular solution are of $O(Re\,r^0)$ and $O(Re\,St\,r^0)$ in this limit. In the outer region it is convenient to define the outer variables $r^* = r/l_\alpha$ or $r^* = r/l_\Omega$ and $r^{**} = r/l_u$. Then the matching of the particular solution with the outer solution implies that the leading-order terms of the latter behave like $O(Re\,r^{*0})$ and $O(Re\,St\,r^{**0})$ for $r^* \to 0$ and $r^{**} \to 0$. If we now assume that the leading-order terms of the complementary solution behave like $Re\,r^n$ and $Re\,St\,r^m$ for $r \to \infty$, re-expressing them in outer variables shows that they must match with terms of $O(Re^{1-n/2}r^{*n})$ and $O((Re\,St)^{1-m/2}r^{**m})$ in the outer expansion. Obviously the latter terms cannot be larger than those of leading order, from which we deduce that $1-n/2 \geqslant 1$ and $1-m/2 \geqslant 1$, i.e. $n \leqslant 0$ and $m \leqslant 0$. Hence the complementary solution behaves like $\boldsymbol{k} + \boldsymbol{k}'O(r^{-1})$ for $r \to \infty$, where $\boldsymbol{k}$ and $\boldsymbol{k}'$ are two constant vectors. However, the corresponding disturbance must vanish at the wall, which, unlike for the Oseen problem in an unbounded flow, requires $\boldsymbol{k} = \boldsymbol{0}$. Since the outer region can provide a correction to the hydrodynamic force only via $\boldsymbol{k}$ (because $\boldsymbol{k} \neq \boldsymbol{0}$ would result in a Stokeslet in the inner expansion), we conclude that provided $\kappa^{-1} < \min(l_u, l_s, l_\alpha, l_\Omega)$ there cannot be any contribution to the force at $O(Re)$ and $O(Re\,St)$ due to the complementary solution. Hence only knowledge of the particular solution is required for obtaining the first-order inertial corrections.

In contrast, as shown by Cox & Hsu (1977), evaluation of second-order inertial effects requires the outer expansion to be considered. To avoid this step, we assume that the latter effects (which are of order Re^2, $Re^2 St$, and $(Re\,St)^2$) are negligibly small compared to leading-order inertial corrections of order Re and $Re\,St$. This is consistent provided the flow conditions satisfy

$$Re \ll 1 \quad \text{and} \quad Re \ll St \ll Re^{-1/2}. \tag{6}$$

In other words, the second of conditions (6) determines the range of St within which leading-order corrections due to unsteadiness and inertia may simply be added without having to consider the possible changes of inertial effects due to temporal acceleration. Note that the last inequality in (6) is equivalent to $Re\,St \ll Re^{1/2}$, a condition more restrictive that the initial condition $Re\,St \ll 1$ required for the leading-order disturbance to be governed by the steady Stokes equation.

4. The reciprocal theorem

We shall obtain the $O(Re)$ and $O(Re\,St)$ contributions to the hydrodynamic force through an application of the reciprocal theorem. For this we first need to consider the auxiliary problem of a spherical drop translating steadily at zero Reynolds number near the wall in a quiescent fluid. Let $\boldsymbol{e}$ be the unit vector along this direction of translation, the orientation of which with respect to the wall is arbitrary. If the velocity of the drop is taken to be unity, the governing equations of the auxiliary problem are

$$\left.\begin{array}{l}
\nabla\cdot\widehat{\boldsymbol{U}} = 0, \quad \nabla\cdot\widehat{\boldsymbol{\Sigma}} = \boldsymbol{0},\\
\nabla\cdot\widehat{\tilde{\boldsymbol{U}}} = 0, \quad \nabla\cdot\widehat{\tilde{\boldsymbol{\Sigma}}} = \boldsymbol{0},\\
\widehat{\boldsymbol{U}} = -\boldsymbol{e} \quad \text{for} \quad x_3 = -1/\kappa,\\
\widehat{\boldsymbol{U}} \to -\boldsymbol{e} \quad \text{for} \quad r \to \infty,\\
\left.\begin{array}{l}
\widehat{\boldsymbol{U}}\cdot\boldsymbol{e}_r = \widehat{\tilde{\boldsymbol{U}}}\cdot\boldsymbol{e}_r = 0\\
\boldsymbol{e}_r \times \widehat{\boldsymbol{U}} = \boldsymbol{e}_r \times \widehat{\tilde{\boldsymbol{U}}}\\
\boldsymbol{e}_r \times (\widehat{\boldsymbol{\Sigma}}\cdot\boldsymbol{e}_r) = \lambda\boldsymbol{e}_r \times (\widehat{\tilde{\boldsymbol{\Sigma}}}\cdot\boldsymbol{e}_r)
\end{array}\right\} \quad \text{for} \quad r = 1,
\end{array}\right\} \tag{7}$$

where $\widehat{\boldsymbol{U}}$ and $\widehat{\boldsymbol{\Sigma}}$ (resp. $\widehat{\tilde{\boldsymbol{U}}}$ and $\widehat{\tilde{\boldsymbol{\Sigma}}}$) are the velocity and stress in the suspending (resp. inner) fluid, respectively, and it must be noticed that Laplace's equation requires the modified pressure inside the drop to be $\hat{\tilde{P}} + 2/Ca$. The above problem was solved up to terms of $O(\kappa^3)$ in MTL using Faxén's technique. This solution is summarized in Appendix A.

The reciprocal theorem has long been used to evaluate inertial and deformation-induced lift forces on drops and particles moving at low Reynolds number in wall-bounded shear flows (see in particular Ho & Leal 1974 and Chan & Leal 1979). More recently, this theorem was shown to also be useful for incorporating unsteady effects in the force experienced by a rigid particle moving at low Reynolds number in an unbounded fluid (Lovalenti & Brady 1993*a*). Moreover, in an appendix to the latter paper, Lovalenti, Brady & Stone showed that it is possible to write down a general form of the reciprocal theorem with no limitation on the magnitude of inertial effects. Lovalenti & Brady (1993*b*) generalized this approach to a spherical drop of arbitrary viscosity. Here we apply a similar procedure in the presence of a wall, starting from (1)–(2) combined with the governing equations of the auxiliary problem (7). Details of the derivation are given in Appendix B. For a spherical drop ($\boldsymbol{n}=\boldsymbol{e}_r$), the final result (B4) is

$$\frac{4}{3}\pi\bar{\rho} Re\, Ste \cdot \frac{\mathrm{d}\boldsymbol{V}_B}{\mathrm{d}t} = \frac{4}{3}\pi \boldsymbol{e} \cdot \left[(\bar{\rho}-1)\boldsymbol{g} + Re \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t'} \right] + \hat{\boldsymbol{F}}_D \cdot \boldsymbol{V}_{S0} + \boldsymbol{S} : \int_{A_B} [2\widehat{\boldsymbol{U}}\boldsymbol{e}_r - \boldsymbol{x}\hat{\boldsymbol{\Sigma}}\cdot\boldsymbol{e}_r]\,\mathrm{d}S$$
$$- Re \int_{V_F} (\widehat{\boldsymbol{U}} + \boldsymbol{e}) \cdot \boldsymbol{f}\,\mathrm{d}V - \bar{\rho} Re \int_{V_B} \hat{\tilde{\boldsymbol{U}}} \cdot \left(St \frac{\partial \tilde{\boldsymbol{U}}}{\partial t} + (\tilde{U} \cdot \nabla)\tilde{\boldsymbol{U}} \right) \mathrm{d}V, \quad (8)$$

where $\hat{\boldsymbol{F}}_D = \int_{A_B} \hat{\boldsymbol{\Sigma}} \cdot \boldsymbol{e}_r\,\mathrm{d}S$ is the drag force on the drop in the auxiliary problem, $\boldsymbol{V}_{S0} = \boldsymbol{V}_B - \boldsymbol{V}|_{\boldsymbol{x}=\boldsymbol{0}}$ is the instantaneous slip velocity at the centre of the drop, and V_B and V_F denote the volume of the drop and the entire volume of fluid surrounding it, respectively. The result (8) is completely general for the class of problems considered here, i.e. a spherical drop moving in a linear flow bounded by a single wall, both fluids being Newtonian. Again we stress that there is no limitation on the magnitude of Re in (8). Moreover, the only condition required on the velocity (resp. stress) disturbance at large distances from the drop is a decay in $\boldsymbol{r}^{-\beta}$ (resp. $r^{-\beta-1}$) with $\beta > 0$; this condition is obviously satisfied however large Re.

To make use of (8), we introduce the strained coordinates $\bar{x}_i = \kappa x_i$ ($i = 1, 3$), $\bar{r} = \kappa r$ and divide the volume V_F into an inner† region V_I and an outer region V_O such that

$$\left.\begin{aligned} V_{\mathrm{I}} &= \{\boldsymbol{r} | 1 \leqslant r < \gamma_0 \kappa^{\chi-1}\}, \\ V_{\mathrm{O}} &= \{\bar{\boldsymbol{r}} | \gamma_0 \kappa^{\chi} < \bar{r} < \infty, -1 \leqslant \bar{x}_3 < \infty\}, \end{aligned}\right\} \quad (9)$$

where γ_0 and χ are two arbitrary constants such that $\gamma_0 = O(\kappa^0)$ and $0 < \chi < 1$. Noting that $\nabla \equiv \kappa \bar{\nabla}$ and $\mathrm{d}^3\boldsymbol{x} \equiv \kappa^{-3}\,\mathrm{d}^3\bar{\boldsymbol{x}}$, the volume integral over V_F in (8) becomes

$$\kappa^{-3} \int_{V_O} (\bar{\hat{\boldsymbol{U}}} + \boldsymbol{e}) \cdot \left(St \frac{\partial \bar{\boldsymbol{u}}}{\partial t} + \kappa(\bar{\boldsymbol{U}} \cdot \bar{\nabla})\bar{\boldsymbol{u}} + \kappa(\bar{\boldsymbol{u}}.\bar{\nabla})\bar{\boldsymbol{V}} \right) \overline{\mathrm{d}V}$$
$$+ \int_{V_I} (\hat{\boldsymbol{U}} + \boldsymbol{e}) \cdot \left(St \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{V} \right) \mathrm{d}V, \quad (10)$$

where $\bar{\hat{\boldsymbol{U}}} + \boldsymbol{e}$ (resp. $\bar{\boldsymbol{u}}$) denotes the velocity field $\hat{\boldsymbol{U}} + \boldsymbol{e}$ (resp. $\boldsymbol{u}$) when expressed in strained coordinates. Near the drop, the multipole expansion of $\hat{\boldsymbol{U}} + \boldsymbol{e}$ may be written in the form $\hat{\boldsymbol{U}} + \boldsymbol{e} = \boldsymbol{e} \cdot (\boldsymbol{I} + \sum_k \boldsymbol{P}^{(k)} \cdot \boldsymbol{M}^k)$ where the $\boldsymbol{M}^{(k)}$ are second-rank tensors depending on the position $\boldsymbol{x}$, and the $\boldsymbol{P}^{(k)}$ are diagonal projectors accounting for the anisotropy of space introduced by the presence of the wall; these projectors depend only of the separation distance between the drop and the wall and on the viscosity ratio λ. Within the drop we may also write $\hat{\tilde{\boldsymbol{U}}} = \boldsymbol{e} \cdot \sum_k \boldsymbol{P}^{(k)} \cdot \tilde{\boldsymbol{M}}^{(k)}$, where the meaning of $\tilde{\boldsymbol{M}}^{(k)}$ is similar to that of $\boldsymbol{M}^{(k)}$. The first two terms of the multipole expansion of

† Note that the meaning of the terms 'inner' and 'outer' used hereinafter differs from that employed in § 3.

$\hat{\boldsymbol{U}}+\boldsymbol{e}$ and $\hat{\hat{\boldsymbol{U}}}$ are given in Appendix A. In the outer region, $\bar{\hat{\boldsymbol{U}}}+\boldsymbol{e}$ may be expanded in the form $\bar{\hat{\boldsymbol{U}}}+\boldsymbol{e}=\boldsymbol{e}\cdot(\sum_k(\boldsymbol{P}_1^{(k)}\cdot\bar{\boldsymbol{M}}_1^{(k)}+\boldsymbol{P}_2^{(k)}\cdot\bar{\boldsymbol{M}}_2^{(k)}))$, where the second-rank tensors $\bar{\boldsymbol{M}}_1^{(k)}$ account for the difference between the disturbance due to the drop and that due to its image located at $\bar{x}_1=\bar{x}_2=0, \bar{x}_3=-2$, and $\bar{\boldsymbol{M}}_2^{(k)}$ is associated with an additional velocity disturbance required to satisfy the no-slip boundary condition on the wall. Since we only plan to obtain the inertial corrections up to terms of $O(Re)$ and $O(Re\,St)$, it turns out that we only need to know the terms corresponding to $k=1$ in the expansion of $\bar{\hat{\boldsymbol{U}}}+\boldsymbol{e}$. The corresponding expressions are given in Appendix A.

Introducing now the resistance tensor $\boldsymbol{R}$ such that $\hat{\boldsymbol{F}}_D=-4\pi R_\mu \boldsymbol{e}\cdot\boldsymbol{R}$ (with R_μ defined below in (13)) and the third-rank tensors $\boldsymbol{R}_M$ and $\boldsymbol{R}_V$ such that $\int_{A_B}\boldsymbol{x}\hat{\boldsymbol{\Sigma}}\cdot\boldsymbol{e}_r\,\mathrm{d}S=\boldsymbol{e}\cdot\boldsymbol{R}_M$ and $\int_{A_B}\hat{\boldsymbol{U}}\boldsymbol{e}_r\,\mathrm{d}S=\boldsymbol{e}\cdot\boldsymbol{R}_V$ we may eliminate the arbitrary vector $\boldsymbol{e}$ from (8). We then obtain the force balance

$$
\begin{aligned}
\frac{4}{3}\pi\bar{\rho}Re\,St\frac{\mathrm{d}\boldsymbol{V}_\mathrm{B}}{\mathrm{d}t} &= \frac{4}{3}\pi\left[(\bar{\rho}-1)\boldsymbol{g}+Re\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t'}\right]-4\pi R_\mu\boldsymbol{P}^{(1)}\cdot\mathbf{V}_{S0}+(2\boldsymbol{R}_V-\boldsymbol{R}_M):\boldsymbol{S}\\
&\quad -Re\kappa^{-3}\int_{V_O}\left(\sum_k\left(\boldsymbol{P}_1^{(k)}\cdot\boldsymbol{M}_1^{(k)}+\boldsymbol{P}_2^{(k)}\cdot\boldsymbol{M}_2^{(k)}\right)\right)\\
&\quad \cdot\left(St\frac{\partial\bar{\boldsymbol{u}}}{\partial t}+\kappa(\bar{\boldsymbol{U}}\cdot\bar{\nabla})\bar{\boldsymbol{u}}+\kappa(\bar{\boldsymbol{u}}\cdot\bar{\nabla})\bar{\boldsymbol{V}}\right)\overline{\mathrm{d}V}\\
&\quad -Re\int_{V_I}\left(\boldsymbol{I}+\sum_k\boldsymbol{P}^{(k)}\cdot\boldsymbol{M}^{(k)}\right)\cdot\left(St\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{U}\cdot\nabla)\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{V}\right)\mathrm{d}V\\
&\quad -\bar{\rho}Re\int_{V_B}\sum_k\boldsymbol{P}^{(k)}\cdot\tilde{\boldsymbol{M}}^{(k)}\cdot\left(St\frac{\partial\tilde{\boldsymbol{U}}}{\partial t}+\left(\tilde{\boldsymbol{U}}\cdot\nabla\right)\tilde{\boldsymbol{U}}\right)\mathrm{d}V. \qquad (11)
\end{aligned}
$$

The left-hand side of (11) and the last term on the right-hand side contain the effects of drop inertia. The first term on the right-hand side is the generalized buoyancy force including the effect of the acceleration of the undisturbed flow, whereas the second one is the quasi-steady Stokes drag. All possible inertial effects due to the flow around the drop are collected in the fourth and fifth terms. There is no need to comment more on these terms at this stage. In contrast the third term in the right-hand side deserves a few comments. This term is due to the presence of a non-zero strain rate in the undisturbed flow but it would be zero in an unbounded flow because the velocity (resp. stress) would then be an even (resp. odd) function of the position on the drop surface. Hence this term results from the interaction of the wall with the rate of strain of the undisturbed flow. We may interpret it as a Faxén correction because it occurs at zero Reynolds number and is due to the non-uniformity of $\boldsymbol{V}$ on the drop surface, as can be seen in (8). In an unbounded flow, Faxén corrections occur only in flows with non-zero curvature (more precisely $\nabla^2\boldsymbol{V}\neq\boldsymbol{0}$) (see e.g. Gatignol 1983; Maxey & Riley 1983; Lovalenti & Brady 1993*a*). Here they occur in a linear flow because the auxiliary velocity field $\hat{\boldsymbol{U}}$ involves a stresslet that does not exist in an unbounded flow (see Appendix A). The corresponding contribution may easily be evaluated up to $O(\kappa^3)$ for the shear flow (3) by using (A7). One finds

$$(2\boldsymbol{R}_V-\boldsymbol{R}_M):\boldsymbol{S}=-\frac{\pi}{2}\alpha R_\mu R_S\left(1+\frac{3}{8}R_\mu\kappa\right)\kappa^2\boldsymbol{e}_1+O(\kappa^4) \qquad (12)$$

with

$$R_\mu = \frac{2+3\lambda}{2(1+\lambda)}, \quad R_S = \frac{2+5\lambda}{2(1+\lambda)}. \tag{13}$$

This expression agrees with that found in MTL using a totally different approach. The corresponding term also appears in studies concerning the motion of rigid particles in wall-bounded flows (e.g. Ho & Leal 1974). This term is responsible for the non-zero slip of neutrally buoyant particles in Couette flow (Halow & Wills 1970).

A point to be noticed in (11) concerns the magnitude of terms $Re\,\mathrm{D}\boldsymbol{V}/\mathrm{D}t'$ and $Re\,St\,\mathrm{d}\boldsymbol{V}_B/\mathrm{d}t$. Since the characteristic velocity scale V_C was chosen with respect to the magnitude of the slip velocity and not that of the absolute velocities $\boldsymbol{V}$ and $\boldsymbol{V}_B$, the above two terms may be of $O(1)$ in certain situations. This may for instance be the case if the flow rotates as a whole like that defined by (4): if the centrifugal force at $\boldsymbol{x}=0$ is much larger than the body force $\boldsymbol{g}$, the motion of the drop relative to the flow is driven by the former force so that $Re\,St\,\|\mathrm{d}\boldsymbol{V}_\mathrm{B}/\mathrm{d}t\| \approx Re\,\|\mathrm{D}\boldsymbol{V}/\mathrm{D}t'\| = -\Omega^2 r_0$ with $r_0 = (x_{10}^2 + x_{20}^2)^{1/2}$, both terms being of $O(1)$ (see §7).

To make use of the reciprocal theorem, we must specify the expression for the velocity disturbances $\boldsymbol{u}$, $\bar{\boldsymbol{u}}$ and $\tilde{\boldsymbol{U}}$ involved in the volume integrals of (11). The $O(Re)$ and $O(Re\,St)$ inertial terms merely result from velocity disturbances satisfying Stokes equations for each particular situation under consideration. The corresponding expressions required to obtain the $O(\kappa^0)$-approximation of the inertial correction are given in Appendix C. The techniques used to evaluate the volume integrals are similar to those described in §6 of MTL to which the interested reader is referred.

5. Effects of temporal acceleration

We start by considering the contribution of unsteady terms in (11), i.e. terms proportional to $\partial\bar{\boldsymbol{u}}/\partial t$, $\partial\boldsymbol{u}/\partial t$ and $\partial\tilde{\boldsymbol{U}}/\partial t$ in the volume integrals of the right-hand side, disregarding for the moment the effects of the nonlinear terms. To express and discuss the result it is convenient to split the slip velocity $\boldsymbol{V}_{S0}(t)$ into its tangential and normal components with respect to the wall, i.e. we write $\boldsymbol{V}_{S0}(t) = \boldsymbol{V}_{S0}^{\parallel}(t) + \boldsymbol{V}_{S0}^{\perp}(t)$ where $\boldsymbol{V}_{S0}^{\perp}(t) = (\boldsymbol{e}_3 \cdot \boldsymbol{V}_{S0}(t))\boldsymbol{e}_3$. Then, evaluating the corresponding volume integrals, the contribution $\boldsymbol{F}_U$ of the temporal acceleration to the hydrodynamic force takes the form

$$\boldsymbol{F}_U = -\pi R_\mu^2 Re\,St \left\{ \kappa^{-1}\left[3\,(1-2I_D)\frac{\mathrm{d}\boldsymbol{V}_{S0}^{\parallel}}{\mathrm{d}t} + (1-2J_D)\frac{\mathrm{d}\boldsymbol{V}_{S0}^{\perp}}{\mathrm{d}t}\right] - \frac{8}{3(2+3\lambda)^2}\left(3+10\lambda+8\lambda^2-\frac{\bar{\rho}}{7}\right)\frac{\mathrm{d}\boldsymbol{V}_{S0}}{\mathrm{d}t}\right\} + O(\kappa). \tag{14a}$$

Using the expressions for I_D and J_D given in (A 5), i.e. $I_D = -\frac{3}{8}R_\mu\kappa$ and $J_D = -\frac{3}{4}R_\mu\kappa$, we may re-express the result (14a) in the form

$$\boldsymbol{F}_\mathrm{U} = -\pi R_\mu^2 Re\,St \left\{ \left[3\kappa^{-1} + \frac{24+140\lambda+306\lambda^2+217\lambda^3}{24(1+\lambda)(2+3\lambda)^2} + \frac{8}{21(2+3\lambda)^2}\bar{\rho}\right]\frac{\mathrm{d}\boldsymbol{V}_{S0}^{\parallel}}{\mathrm{d}t} + \left[\kappa^{-1} - \frac{24+92\lambda+90\lambda^2+13\lambda^3}{12(1+\lambda)(2+3\lambda)^2} + \frac{8}{21(2+3\lambda)^2}\bar{\rho}\right]\frac{\mathrm{d}\boldsymbol{V}_{S0}^{\perp}}{\mathrm{d}t}\right\} + O(\kappa). \tag{14b}$$

Given the assumptions made in §§2 and 3, this result is valid provided $Re\,St \ll 1$ and $\kappa^{-1} \ll (Re\,St)^{-1/2}$. According to (14$b$) the component of $\boldsymbol{F}_U$ parallel to the wall is always in the opposite direction to the parallel component of the acceleration $\mathrm{d}\boldsymbol{V}_{S0}^{\parallel}/\mathrm{d}t$.

The same property holds for the leading-order term of the normal component but the following term has a positive contribution that lowers the net normal component of $\boldsymbol{F}_U$. Note that the coefficient of the $O(\kappa^{-1})$ term in (14) is three times larger for an acceleration parallel to the wall than for an acceleration in the normal direction.

To gain some insight into the physical origin of the contributions involved in the above result, it is of particular interest to compare (14*a*) with the result obtained in an unbounded flow by Lovalenti & Brady (1993*b*) for a spherical drop experiencing an unsteady motion satisfying the condition $Re\,St \ll 1$. In the present notation their equations (61)–(62) become

$$\boldsymbol{F}_U^{LB}(t) = \boldsymbol{F}_H(t) - \pi R_\mu^2 Re\,St \left\{ \frac{16}{9} R_\mu - \frac{8}{3(2+3\lambda)^2}\left(3 + 10\lambda + 8\lambda^2 - \frac{\bar{\rho}}{7}\right)\right\} \frac{\mathrm{d}\boldsymbol{V}_{S0}}{\mathrm{d}t} \quad (15)$$

with (equation (6.15) of Lovalenti & Brady 1993*a* and (55) of Lovalenti & Brady 1993*b*),

$$\boldsymbol{F}_H(t) = -2\pi^{1/2}(\mathrm{Re}\,St)^{1/2} R_\mu^2 \int_{-\infty}^{t} \left[\frac{2}{3}\boldsymbol{V}_{S0}(t) - \frac{1}{A^2}\left(\frac{\pi^{1/2}}{2A}\mathrm{erf}(A) - \mathrm{e}^{-A^2}\right) \times \left(\boldsymbol{V}_{S0}^{\rightarrow}(s) - \frac{1}{2}\boldsymbol{V}_{S0}^{\uparrow}(s)\right) - \mathrm{e}^{-A^2}\boldsymbol{V}_{S0}^{\uparrow}(s)\right] \frac{\mathrm{d}s}{(t-s)^{3/2}}, \quad (16)$$

where

$$A = \frac{1}{2}\left(\frac{Re}{St(t-s)}\right)^{1/2} \left\| \int_s^t \boldsymbol{V}_{S0}(u)\,\mathrm{d}u \right\|,$$

and $\boldsymbol{V}_{S0}^{\rightarrow}$ (resp. $\boldsymbol{V}_{S0}^{\uparrow}$) denotes the contribution of the slip velocity parallel (resp. perpendicular) to the displacement vector $\int_s^t \boldsymbol{V}_{S0}(u)\,\mathrm{d}u$. The force $\boldsymbol{F}_H(t)$, called the 'unsteady Oseen force' by Lovalenti & Brady, is the long-time counterpart of the familiar Basset–Boussinesq history force. It comes from the Oseen wake of the particle, i.e. from the wake region located at downstream distance l from the particle such that $l > Re^{-1}$. In this region, advection is more efficient in transporting vorticity downstream than viscous diffusion, which makes the contribution of history effects to the total hydrodynamic force smaller than predicted by the Basset–Boussinesq expression.

The leading-order term of (14) is proportional to κ^{-1} but cannot grow without bound as the separation distance between the drop and the wall increases. According to the discussion of § 3, the maximum separation for which our result may be qualitatively valid is $\kappa^{-1} \sim (Re\,St)^{-1/2}$, corresponding to a leading-order contribution to the force of $O(Re\,St)^{1/2}$. As shown by (16), this is precisely the order of magnitude of $\boldsymbol{F}_H(t)$. If we note in addition that the leading-order term in (14*a*) and the force $\boldsymbol{F}_H(t)$ are both proportional to R_μ^2, we have a strong indication that the $O(\kappa^{-1})$ force in (14*a*) is what is left of the 'unsteady Oseen force' $\boldsymbol{F}_H(t)$ in the presence of a wall. This matching is expected on physical grounds. Terms of $O(\kappa^{-1})$ in (14) and $\boldsymbol{F}_H(t)$ in (15) are the leading-order contributions to the net force due to the same physical cause, i.e. temporal acceleration. One is obtained through a regular expansion procedure because the wall lies in the inner region of the disturbance, while the other is found in the absence of the wall through a singular perturbation analysis. Nevertheless, as any hydrodynamical process evolves smoothly between these two limits, there is an intermediate situation, that where the wall is located in the outer region of the disturbance, for which the two scalings must match. As we have seen that the $O(\kappa^{-1})$ term in (14) becomes of the same order of magnitude as the force $\boldsymbol{F}_H(t)$ for $\kappa^{-1} \sim (Re\,St)^{-1/2}$, i.e. when the wall lies at the outer limit of the Stokes region, we may infer that for larger separations, for which the wall lies in the outer region ($\kappa \ll (Re\,St)^{1/2}$), the unsteady Oseen force remains of order $(Re\,St)^{1/2}$ whatever κ. Identifying the unsteady Oseen force in an unbounded flow as the force that matches the $O(\kappa^{-1})$ term of (14) for $\kappa^{-1} \sim (Re\,St)^{-1/2}$ leads us to conclude that the latter term is not a force created by the wall (we shall find such forces in the next section), but is simply an effect already present in an unbounded flow and altered by the wall.

The remarkable difference between the two extreme situations corresponding to (14) and (15), respectively, is that in the presence of a wall the leading-order contribution to $\boldsymbol{F}_U(t)$ is directly proportional to the 'instantaneous' acceleration $(\mathrm{d}\boldsymbol{V}_{S0}/\mathrm{d}t)(t)$ and does not involve any convolution integral of past accelerations. We may interpret this as a drastic reduction of the flow memory due to the screening-out of the outer region by the wall, the reason for which is easily understood by noting that the disturbance created by the particle requires a dimensionless time of order $(\kappa Re)^{-1}$ (resp. Re^{-2}) to reach the wall (resp. the Oseen wake), and we are considering situations in which $\kappa^{-1} \ll Re^{-1}$. For instance, after an abrupt change in the slip velocity at $t = t_0$, the present results show that there is no unsteady contribution left to the hydrodynamic force for $t \geqslant t_0 + O(Re\,St)^{-1}$ (keeping in mind that our results do not apply within the time interval $[t_0, t_0 + O(1)]$). Moreover, the corresponding force is clearly reduced as the drop approaches the wall since its magnitude decreases from $O(Re\,St)^{1/2}$ for $\kappa^{-1} \sim (Re\,St)^{-1/2}$ to $O(Re\,St)$ for $\kappa \to 1$. The reason for this is that the maximum effect of the wall on the velocity disturbance is felt at distances of $O(\kappa^{-1})$ (because the wall is located at κ^{-1} from the particle centre), whereas the unsteady Oseen force comes from the wake region located at a distance of $O(Re^{-1})$ from the particle. Thus the region of the flow where the maximum diturbance due to effects of temporal acceleration arises lies closer to the particle in the present wall-bounded situation, and the smaller the distance between the drop and the location of the maximum disturbance, the smaller the volume of fluid affected by these effects. It is also interesting to note that, provided $\mathrm{d}\boldsymbol{V}_{S0}^{\perp}/\mathrm{d}t$ is unchanged, the normal component of the force (14) is left unchanged if the direction of $\boldsymbol{V}_{S0}^{\perp}$ is reversed. For instance, at a given distance from the wall, a drop moving towards the wall ($\boldsymbol{V}_{S0}^{\perp} < \boldsymbol{0}$) and decelerating at a rate $\mathrm{d}\boldsymbol{V}_{S0}^{\perp}/\mathrm{d}t = \beta \boldsymbol{e}_3$, with $\beta > 0$, experiences the same unsteady force as if it recedes from the wall ($\boldsymbol{V}_{S0}^{\perp} > \boldsymbol{0}$) and accelerates at the same rate $\beta > 0$, which may seem surprising at first glance. Again, the reason is that the force is mainly due to wall-induced modifications to the flow structure at distances of $O(\kappa^{-1})$, and this corresponds to a region where the flow is still reversible.

In (15) the remaining terms in the right-hand side are due on the one hand to the added-mass force $-\frac{2}{3}\pi\,\mathrm{d}\boldsymbol{V}_{S0}/\mathrm{d}t$ and on the other hand to the difference between the history force acting on the drop (that depends on both λ and $\bar{\rho}$) and a second memory integral coming from the Oseen wake, the role of which is to cancel the leading-order contribution of the history force that diverges at long time if $\mathrm{d}\boldsymbol{V}_{S0}/\mathrm{d}t$ remains constant (see Lovalenti & Brady 1993*a*). The same contributions (with the Oseen wake replaced by the region located at distances $O(\kappa^{-1})$ from the drop) exist in (14) as can be shown by comparing (14*a*) and (15). The $O(\kappa^0)$ term on the right-hand side of (14*b*) may then be interpreted as the sum of the added-mass force arising from temporal acceleration and the non-diverging part of the history force (both taken in the long-time limit where $\boldsymbol{V}_{S0}$ does not vary significantly over an $O(1)$ time interval), plus a wall-induced contribution from memory effects acting at distances $O(\kappa^{-1})$ and whose leading-order term cancels the diverging part of the history force. This wall-induced contribution (corresponding to terms proportional to I_D and J_D in (14*a*)) is the counterpart of the term $\frac{16}{9}R_\mu$ in (15), as one can detect by noting that both terms are proportional to R_μ^3. The remaining $O(\kappa^0)$ terms in (14*a*) are identical to those of (15), i.e. they are not modified by the wall at the present order of approximation. This is because these terms come from the drop and its immediate surroundings (the region V_I of (11)).

To conclude our discussion of (14), it is important to stress that the corresponding force must not be misinterpreted. Since this force is proportional to the components of $\mathrm{d}\boldsymbol{V}_{S0}/\mathrm{d}t$, it could be tempting to interpret it entirely as an added-mass force and one would then conclude that added-mass effects depend on both $\bar{\rho}$ and λ. However, this is by no means the case. Added-mass effects due to temporal acceleration are usually defined as those due to the instantaneous response of the particle to a change in the slip velocity, 'instantaneous' meaning that these effects operate on a characteristic time scale much smaller than any of the other time scales involved in the hydrodynamic processes. Here this means that the characteristic time scale of added-mass effects is $o(1)$, much smaller than the viscous time scale $\rho R^2/\mu$ by which all time scales were normalized. With this definition in mind, it may be shown that added-mass effects in an unbounded flow do not depend on any of the parameters Re, St, $\bar{\rho}$ or λ, and this result is not limited to low Reynolds numbers (see e.g. the analysis presented by Mougin & Magnaudet 2002). On the other hand, history effects are a direct consequence of unsteady viscous diffusion, so that their characteristic time scale is $O(1)$. Since our analysis is restricted to variations of $\boldsymbol{V}_{S0}$ with a characteristic time scale of $O((Re\,St)^{-1})$ with $Re\,St \ll 1$, added-mass effects and history effects appear to be both associated with infinitely short (i.e. $o(1)$) time scales. This is why they both involve the instantaneous acceleration $\mathrm{d}\boldsymbol{V}_{S0}/\mathrm{d}t$, 'instantaneous' meaning now that this acceleration does not vary significantly over $o(Re\,St)^{-1}$ time scales. So, in order to avoid any confusion with the added-mass force, the force $\boldsymbol{F}_U$ in (14) which contains all possible effects of temporal acceleration may be globally referred to as the 'long-time reaction to temporal acceleration', following Lovalenti & Brady (1993*a*).

6. Slip-induced and shear-induced lift

We now turn to the contributions to the total force coming from the nonlinear terms of the three volume integral in (11), and begin by considering the situation where the undisturbed flow is given by (3). Corresponding results were obtained by Cox & Hsu (1977) and Lovalenti in an Appendix to the article of Cherukat & McLaughlin (1994) for a rigid sphere, and by MTL for a drop of arbitrary viscosity. They may be recast in the form

$$\boldsymbol{F}_{\mathrm{L}} = \frac{\pi}{4} Re R_\mu^2 \left\{ \left[(\boldsymbol{V}_{S0}^{\parallel}(t) \cdot \boldsymbol{V}_{S0}^{\parallel}(t)) + \tfrac{11}{18}\alpha^2 \frac{R_S}{R_\mu} \right] \boldsymbol{e}_3 - \tfrac{5}{2}\alpha(\kappa^{-1} - E_0(\lambda))\,(\boldsymbol{V}_{S0}(t)\cdot \boldsymbol{e}_3)\,\boldsymbol{e}_1 \right. \\ \left. - \tfrac{11}{6}\alpha(\kappa^{-1} + D_0(\lambda))\,(\boldsymbol{V}_{S0}(t)\cdot\boldsymbol{e}_1)\,\boldsymbol{e}_3 \right\} + O(\kappa), \quad (17)$$

with

$$D_0(\lambda) = \frac{3960 + 12444\lambda + 14826\lambda^2 + 6645\lambda^3}{880(2+3\lambda)^2(1+\lambda)},$$

$$E_0(\lambda) = \frac{4840 + 14100\lambda + 11934\lambda^2 + 2191\lambda^3}{1200(2+3\lambda)^2(1+\lambda)},$$

R_S being defined in (13). Note that, owing to symmetry, none of the terms present in (17) involves the drop density (see MTL for further remarks on this point). The first term on the right-hand side of (17) is a lift force arising from the component of the slip velocity parallel to the wall. This force makes the drop migrate away from the wall; it is responsible for the migration of non-neutrally buoyant particles rising or falling near a vertical wall in a quiescent fluid. Experimental confirmation of the above prediction in the two limit cases $\lambda \to \infty$ and $\lambda \to 0$ was provided by the experiments of Vasseur & Cox (1977) and Cherukat & McLaughlin (1990) with rigid particles, and those of Takemura *et al.* (2002) with bubbles rising in silicone oil, respectively. The second term on the right-hand side of (17) is also a force normal to the wall and directed away from it. It results from the interaction of the stresslet involved in the velocity disturbance (see (C1a)) with the wall, and may provide the dominant contribution to (17) for very large shear or neutrally buoyant particles.

The third term on the right-hand side of (17) is a lift force parallel to the wall (more precisely, parallel to the streamlines of the base flow). It occurs when the drop moves across the shear (here in the x_3-direction), and tends to maintain the drop behind the flow if $\boldsymbol{V}_{S0}(t) \cdot \boldsymbol{e}_3$ is positive. A similar effect exists in an unbounded flow. It was first evaluated in the low-Re limit by Harper & Chang (1968), and later reconsidered by Hogg (1994) and Miyazaki, Bedeaux & Bonet Avalos (1995). We can use the same arguments as in § 5 to show that the leading-order contribution to the term under consideration in (17) is what is left of the Harper & Chang lift force in the presence of a wall. That is, considering the upper limit of the separation distance for which the present prediction may qualitatively be valid, i.e. $\kappa^{-1} \sim (\alpha Re)^{-1/2}$ (see § 3), we see that the leading-order contribution to the force becomes of $O(\alpha Re)^{1/2}$ at such distances; this is indeed the order of magnitude of the force in the unbounded case. Moreover, following the general argument developed by Legendre & Magnaudet (1997), it can be shown that in an unbounded linear flow the magnitude of the shear-induced lift force on a spherical drop of arbitrary viscosity is R_μ^2 times that of the lift force on a rigid sphere of the same radius. The contribution under consideration in (17) is also proportional to R_μ^2, so that both scaling arguments show that it behaves like the Harper & Chang lift force.

The last term on the right-hand side of (17) exists when there is a non-zero slip velocity along the streamlines of the shear flow. Using arguments similar to those detailed above, it turns out that this force is the counterpart of the well-known Saffman lift force (Saffman 1965) in an unbounded shear flow. The evolution of this force from the unbounded case considered by Saffman to the near-wall situation studied here via the intermediate case where the wall lies in the outer region of the disturbance has been studied in great detail by McLaughlin (1993) In particular, figure 4 of McLaughlin's paper allows us to appreciate how the $O(\kappa^{-1}\alpha Re)$ near-wall force matches the $O(\alpha Re)^{1/2}$ Saffman lift force for separations such that $\kappa(\alpha Re)^{-1/2} = O(1)$. If the drop leads the flow, i.e. $\boldsymbol{V}_{S0}(t) \cdot \boldsymbol{e}_1$ is positive, this force pushes it toward the low-velocity side of the undisturbed flow, i.e. toward the wall. Note that, even though the slip velocity $\boldsymbol{V}_{S0}$ in (17) is formally allowed to vary in time, conditions (6) are assumed to hold. Hence there is no correction to the slip-induced or shear-induced lift force due to unsteadiness at the present level of approximation. In other words, (17) provides only a quasi-steady approximation of the slip- and shear-induced lift forces; explicit evaluation of effects of unsteadiness on these forces (like the unsteady Saffman lift force on a sphere oscillating in an unbounded shear flow computed by Asmolov & McLaughlin 1999) would require contributions of $O(Re^2St)$ to be considered.

It is worth noting that the connection between the two near-wall contributions just discussed and their counterpart in an unbounded flow goes beyond scaling arguments.

By this we mean that in both situations the corresponding forces are negative for positive values of $\boldsymbol{V}_{S0}(t)\cdot\boldsymbol{e}_1$ and $\boldsymbol{V}_{S0}(t)\cdot\boldsymbol{e}_3$, and that the magnitude of the component of the lift force parallel to the shear is larger than that of the component normal to it. In an unbounded flow the ratio of these two lift forces is about 1.8 according to the results of Harper & Chang (1968) and Hogg (1994), while the near-wall situation yields a ratio 15/11.

Also, it is perhaps worth stressing the following point, which is not specific to the wall-bounded situation but is frequently overlooked. In the limit of infinite Reynolds number, weak shear and negligible effects of unsteadiness, Auton (1987) showed that the shear-induced lift force on a sphere takes the form (in the present notation) $\boldsymbol{F}_L = \frac{2}{3}\pi(\boldsymbol{\omega}\times\boldsymbol{V}_{S0})$, where $\boldsymbol{\omega}=\nabla\times\boldsymbol{V}$. In the present case $\boldsymbol{\omega}=\alpha\boldsymbol{e}_2$, from which we deduce in this limit that

$$\boldsymbol{F}_L = \tfrac{2}{3}\pi\alpha[(\boldsymbol{V}_{S0}\cdot\boldsymbol{e}_3)\boldsymbol{e}_1 - (\boldsymbol{V}_{S0}\cdot\boldsymbol{e}_1)\boldsymbol{e}_3].$$

Comparing with (17) and with the corresponding expressions of Harper & Chang (1968), Hogg (1994) and Miyazaki *et al.* (1995), we see that the component of the force normal to the streamlines retains the same sign in the two limits $Re \ll 1$ and $Re\to\infty$, while the component parallel to $\boldsymbol{e}_1$ has an opposite sign in these two limits. This difference clearly shows that the low-Re lift force is not expressible only in terms of the vorticity of the undisturbed flow. Rather, vorticity and strain are both responsible for the lift force, and particles moving in pure strain flows may also experience a lift force (see Drew 1978; Bedeaux & Rubi 1987; and Perez-Madrid, Rubi & Bedeaux 1990).

To conclude this section it is of interest to discuss how effects of advection and temporal acceleration combine in the particular situation where there is a constant and non-zero normal slip velocity $\boldsymbol{V}_{S0}^{\perp}$; as shown by MTL this occurs at leading order for a buoyant drop moving near a horizontal wall. Let us first re-write the force balance (11) in terms of the slip velocity $\boldsymbol{V}_{S0}$. After some transformations we obtain

$$\tfrac{4}{3}\pi\bar{\rho}Re\,St\left[\frac{\mathrm{d}\boldsymbol{V}_{S0}}{\mathrm{d}t}+\alpha(\boldsymbol{V}_{S0}\cdot\boldsymbol{e}_3)\boldsymbol{e}_1\right] = \tfrac{4}{3}\pi(\bar{\rho}-1)\left[\boldsymbol{g}-Re\,St\frac{\mathrm{d}\boldsymbol{V}_W}{\mathrm{d}t}\right] \\ -4\pi R_\mu \boldsymbol{P}^{(1)}\cdot\boldsymbol{V}_{S0} + (2\boldsymbol{R}_V-\boldsymbol{R}_M):\boldsymbol{S}+\boldsymbol{F}_U+\boldsymbol{F}_L, \quad (18)$$

where $\boldsymbol{F}_U$ and $\boldsymbol{F}_L$ are given by (14*b*) and (17), respectively. Then, in the directions parallel to the wall two limit cases may occur. If there is no body force parallel to the wall (i.e. the first term on the right-hand side of (18) has only a component perpendicular to the wall), the drop moves so as to maintain at all time a small parallel component of the slip velocity $\boldsymbol{V}_{S0}^{\parallel}$ (figure 2a). Then $\mathrm{d}\boldsymbol{V}_{S0}^{\parallel}/\mathrm{d}t$ is small and so is the component of $\boldsymbol{F}_U$ parallel to the wall. The leading-order contribution $\boldsymbol{V}_{S0}^{\parallel(0)}$ to $\boldsymbol{V}_{S0}^{\parallel}$ then results from a balance between the Faxén force (12) and the quasi-steady drag, yielding $\boldsymbol{V}_{S0}^{\parallel(0)}(t) = -\frac{1}{8}\alpha R_S\kappa^2(t)\boldsymbol{e}_1 + O(\kappa^4)$. Noting that $St=1$ because the time scale characterizing the variation of $\boldsymbol{V}_B$ is the advective time scale and using (17) and (A3*a*), (18) can be expanded in powers of αRe. Dividing the result by $\pi\alpha Re$, the $O(\alpha Re)$ force balance is found to be

$$\tfrac{4}{3}\bar{\rho}\big(\boldsymbol{V}_{S0}^{(0)}\cdot\boldsymbol{e}_3\big)\boldsymbol{e}_1 = -4R_\mu\big(1+\tfrac{3}{8}R_\mu\kappa\big)\boldsymbol{V}_{S0}^{\parallel(\alpha Re)} - \tfrac{5}{8}R_\mu^2(\kappa^{-1}-E_0(\lambda))\big(\boldsymbol{V}_{S0}^{(0)}\cdot\boldsymbol{e}_3\big)\boldsymbol{e}_1, \quad (19)$$

where $\boldsymbol{V}_{S0}^{\parallel(\alpha Re)}$ is the $O(\alpha Re)$ contribution to $\boldsymbol{V}_{S0}^{\parallel}$.

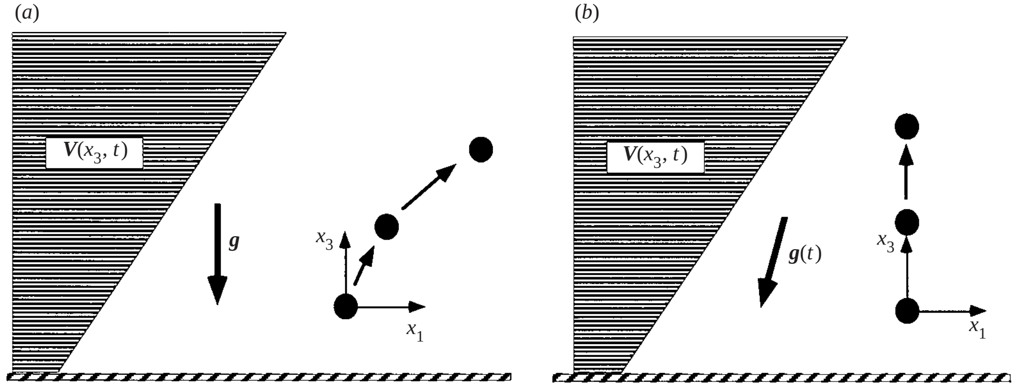

FIGURE 2. Two limit situations for a drop moving in a wall-bounded shear flow with a slip velocity normal to the wall: (*a*) no body force acts parallel to the wall; (*b*) a parallel body force prevents the drop from moving parallel to the wall.

A different case is encountered if the body force $\boldsymbol{g}$ prevents any motion of the drop parallel to the wall. Then $\mathrm{d}\boldsymbol{V}_B^{\|}/\mathrm{d}t = \boldsymbol{0}$ and $\mathrm{d}\boldsymbol{V}_{S0}^{\|(0)}/\mathrm{d}t = -\alpha(\boldsymbol{V}_{S0}^{(0)} \cdot \boldsymbol{e}_3)\boldsymbol{e}_1$, because the fluid velocity 'seen' by the drop varies as the drop moves away from or toward the wall (figure 2*b*). Adding the right-hand sides of (14*b*) and (17) and assuming that the wall is at rest, we now find that the $O(\alpha Re)$ force balance is

$$\boldsymbol{0} = \tfrac{4}{3}(\bar{\rho}-1)\boldsymbol{g}^{\|(\alpha Re)} + R_\mu^2\left[\tfrac{19}{8}\kappa^{-1} + \frac{6760+25300\lambda+36414\lambda^2+19551\lambda^3}{1920(1+\lambda)(2+3\lambda)^2} + \frac{8}{21(2+3\lambda)^2}\bar{\rho}\right]\left(\boldsymbol{V}_{S0}^{(0)}\cdot\boldsymbol{e}_3\right)\boldsymbol{e}_1, \quad (20)$$

where $\boldsymbol{g}^{\|(\alpha Re)}$ is the $O(\alpha Re)$ contribution to the body force $\boldsymbol{g}$ (which depends on time) required to maintain a zero parallel component of the drop velocity.

7. Lift in a solid-body rotation flow

We now consider the situation where the wall rotates at a constant rate Ω about a point $\boldsymbol{x}_\Omega$ whose coordinates with respect to the centre of the drop are $\boldsymbol{x}_\Omega = -(x_{10}(t), x_{20}(t), 1/\kappa(t))$. The corresponding undisturbed flow is defined by (4).

After evaluating the volume integrals on the right-hand side of (11), we find that the counterpart of (17) is

$$\boldsymbol{F}_L = \frac{\pi}{4}Re\left\{R_\mu^2(\boldsymbol{V}_{S0}^{\|}(t)\cdot\boldsymbol{V}_{S0}^{\|}(t))\boldsymbol{e}_3 + \Omega\left[\tfrac{27}{16}R_\mu^2\left(\kappa^{-1}+\tfrac{3}{4}R_\mu\right)+\tfrac{16}{3}\right]\boldsymbol{e}_3\times\boldsymbol{V}_{S0}(t)\right\} + O(\kappa). \quad (21)$$

Again the first term within curly brackets is the lift force due to the component of the slip velocity parallel to the wall. The second term is a lift force parallel to the wall that results from the interaction of the rotation and the slip velocity. This force deflects the trajectory of the drop out of the plane perpendicular to the wall that contains the primary slip velocity. Such a contribution also exists in an unbounded flow. It was first evaluated correctly in the low-*Re* limit (for a rigid particle) by Gotoh (1990) and later by Miyazaki (1995). Defining the Taylor number $Ta = \Omega Re$ and considering again the upper limit where (21) may be qualitatively valid, i.e. $\kappa^{-1} \sim (Ta)^{-1/2}$ (see § 3), we see that the leading-order contribution to the force becomes of $O((Ta)^{1/2})$ at such distances, and this is indeed the order of magnitude of the force in an unbounded flow. Moreover, we note that in (21) the numerical factor in front of $\boldsymbol{e}_3 \times \boldsymbol{V}_{S0}$ is positive, as it is in the expression found by Gotoh (1990) and Miyazaki (1995) (they obtained $\boldsymbol{F}_L = 3\sqrt{2}(19 - 9\sqrt{3})/280(Ta)^{1/2}\boldsymbol{e}_3 \times \boldsymbol{V}_{S0}$, and this result can again be generalized to a drop of arbitrary viscosity by multiplying the right-hand side by R_μ^2, following Legendre & Magnaudet 1997). Hence the leading-order term in (21) appears to be what is left of Gotoh's lift force near a wall. Similarly to what we found for the long-time reaction to temporal acceleration and for the shear-induced lift force, the rotation-induced lift force is proportional to κ^{-1}. Thus its magnitude decreases as the separation distance decreases, which may be interpreted as a wall-induced reduction of inertial effects existing in unbounded flow. Note that the last $O(\kappa^0)$ term on the right-hand side of (21) is purely inertial since it does not depend on λ.

To analyse the resulting hydrodynamic force experienced by the drop, we first re-write (11) in terms of the slip velocity. After transforming the fluid and drop accelerations, we obtain

$$\tfrac{4}{3}\pi\bar{\rho} Re\, St\left[\frac{\mathrm{d}\boldsymbol{V}_{S0}}{\mathrm{d}t} + \Omega \boldsymbol{e}_3 \times \boldsymbol{V}_{S0}\right]$$
$$= \tfrac{4}{3}\pi(\bar{\rho}-1)\left[\boldsymbol{g} + Re\, St\left(\Omega^2 \boldsymbol{r}_0 - \frac{\mathrm{d}\boldsymbol{V}_\Omega}{\mathrm{d}t}\right)\right] - 4\pi R_\mu \boldsymbol{P}^{(1)} \cdot \boldsymbol{V}_{S0} + \boldsymbol{F}_U + \boldsymbol{F}_L, \quad (22)$$

where $\boldsymbol{V}_\Omega$ is the translational velocity of the wall and $\boldsymbol{r}_0(t) = x_{10}(t)\boldsymbol{e}_1 + x_{20}(t)\boldsymbol{e}_2$. In an unbounded flow, particle trajectories resulting from the creeping-flow approximation of (22) (i.e. considering only buoyancy, centrifugal acceleration and Stokes drag) have been studied by Annamalai & Cole (1986) and Roberts, Kornfeld & Fowlis (1991) (see also the review by Bush, Stone & Tanzosh 1994 where finite-Ta effects are discussed). In what follows we consider that the direction of the body force $\boldsymbol{g}$ does not vary in time; hence, since $\boldsymbol{g}$ and $\mathrm{d}\boldsymbol{V}_\Omega/\mathrm{d}t$ play a similar role in (22), we ignore the latter. Then two limit situations may occur, depending on the relative magnitude of the parallel component $\boldsymbol{g}^\parallel$ of $\boldsymbol{g}$ and the centrifugal acceleration $Re\, St\Omega^2\boldsymbol{r}_0$.

Let us first examine the situation where $\|\boldsymbol{g}^\parallel\|$ is much larger than $Re\, St\Omega^2\|\boldsymbol{r}_0\|$ (figure 3a). This is for instance the case encountered when a buoyant drop crosses a vortex core with a horizontal axis. Then the leading-order contribution $\boldsymbol{V}_{S0}^{\parallel(0)}$ to $\boldsymbol{V}_{S0}^\parallel$ is parallel or antiparallel to $\boldsymbol{g}^\parallel$ (depending on the sign of $\bar{\rho}-1$), and $\mathrm{d}\boldsymbol{V}_{S0}^{\parallel(0)}/\mathrm{d}t$ is zero because the leading-order motion of the particle merely results from a balance between the drag force and the constant body force (except in the possible transient stages of the motion), implying $\boldsymbol{F}_U \approx \boldsymbol{0}$. Using (21) and noting again that $St = 1$ because the time scale characterising the variations of $\boldsymbol{V}_B$ is the advective time scale, we may expand (22) in powers of the Taylor number. After dividing by πTa, the $O(Ta)$ force balance in the fixed direction parallel to the wall and perpendicular to $\boldsymbol{V}_{S0}^{\parallel(0)}$ takes the form

$$\left[\tfrac{27}{64}R_\mu^2\left(\kappa^{-1} + \tfrac{3}{4}R_\mu\right) + \tfrac{4}{3}(1-\bar{\rho})\right]\boldsymbol{e}_3 \times \boldsymbol{V}_{S0}^{(0)} - 4R_\mu\left(1 + \tfrac{3}{8}R_\mu\kappa\right)\boldsymbol{V}_{S0}^{(Ta)} = \boldsymbol{0}, \quad (23)$$

where $\boldsymbol{V}_{S0}^{(Ta)}$ is the $O(Ta)$-correction to the slip velocity.

Now let us examine the opposite situation corresponding to $Re\, St\Omega^2\|\boldsymbol{r}_0\| \gg \|\boldsymbol{g}^\parallel\|$ (figure 3b). This is typically the case for a drop moving in a centrifuge. Then the primary slip velocity is parallel or antiparallel to $\boldsymbol{r}_0$ (see the remark concerning

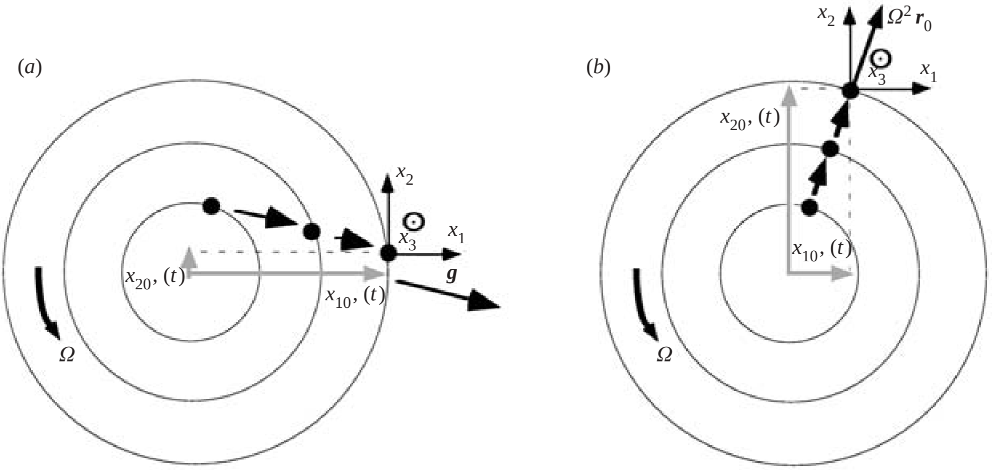

FIGURE 3. Two limit situations for a drop moving in a wall-bounded solid-body rotation flow: (*a*) the drop motion is driven by a body force with a constant direction; (*b*) the drop motion is driven by the centrifugal force.

the scaling of the accelerations at the end of §4), so that its projections along the translating directions x_1, x_2, x_3 are time-dependent with $\mathrm{d}\boldsymbol{V}_{S0}^{\|(0)}/\mathrm{d}t = \Omega \boldsymbol{e}_3 \times \boldsymbol{V}_{S0}^{\|(0)}$. Hence in addition to the 'quasi-steady' component of the lift force parallel to the wall given by (21), say $\boldsymbol{F}_L^{\|}$, there is a contribution $\boldsymbol{F}_U^{\|}$ provided by the force $\boldsymbol{F}_U$ given by (14*b*). The total lift force parallel to the wall is then

$$\boldsymbol{F}_U^{\|} + \boldsymbol{F}_L^{\|} = -\pi Ta \left[R_\mu^2 \left(\frac{165}{64}\kappa^{-1} + \frac{-408 + 212\lambda + 6462\lambda^2 + 7327\lambda^3}{1536(1+\lambda)(2+3\lambda)^2} + \frac{8}{21(2+3\lambda)^2}\bar{\rho} \right) - \frac{4}{3} \right] \boldsymbol{e}_3 \times \boldsymbol{V}_{S0}^{(0)} + O(\kappa). \quad (24)$$

From (22) and (24) we conclude that the force balance in the time-dependent direction parallel to the wall and perpendicular to $\boldsymbol{V}_{S0}^{\|(0)}$ (i.e. the azimuthal direction) is

$$-\tfrac{2}{3}\bar{\rho}\boldsymbol{e}_3 \times \boldsymbol{V}_{S0}^{(0)} + \frac{\boldsymbol{F}_U^{\|} + \boldsymbol{F}_L^{\|}}{4\pi Ta} - R_\mu \left(1 + \tfrac{3}{8} R_\mu \kappa\right) \boldsymbol{V}_{S0}^{(Ta)} = \boldsymbol{0}. \quad (25)$$

In an unbounded flow, the counterpart of (24) was evaluated by Herron, Davis & Bretherton (1975) who found $\boldsymbol{F}_U^{\|} + \boldsymbol{F}_L^{\|} = -\frac{3}{5}(Ta)^{1/2}\boldsymbol{e}_3 \times \boldsymbol{V}_{S0}$ (once again this result may be generalized to a drop of arbitrary viscosity by multiplying the right-hand side by R_μ^2). Note that the sign of the leading-order term in (24) differs from that of the leading-order term in (21), as in the two expressions found by Gotoh (1990) and Herron *et al.* (1975), respectively. Consequently, as we have already seen for a pure shear flow, the presence of the wall reduces the strength of the lift force compared to the unbounded situation but does not change its sign. Moroever, the magnitude of the force determined by Herron *et al.* (1975) is about twelve times that found by Gotoh (1990), which is reflected in the present near-wall results since the lift force given by (24) is about six times larger than that corresponding to (21).

The connection between the result of Gotoh (1990) and of Herron *et al.* (1975) was carefully investigated by Miyazaki (1995), who pointed out that the difference is due to memory effects. Comparing (24) with the term proportional to Ω in (21) leads us to the same conclusion, as the difference comes from $\boldsymbol{F}_U^{\|}$ and we have seen that the leading-order contribution to $\boldsymbol{F}_U$ comes from history effects. As also pointed out by Miyazaki (1995), these two problems involving inertial effects on a particle moving in a rotating flow provide a clear illustration of the limitations of the 'principle' of material frame-indifference, as (*a*) the force experienced by the particle is found to be affected by a solid-body rotation of the undisturbed flow in both cases, and (*b*) the functional relation giving the force as a function of the flow characteristics depends on the motion of the observer, as it is clear that (24) differs from the rotation-induced contribution in (21).

8. Concluding remarks

Let us first summarize the main results of the present investigation. Using the reciprocal theorem, we showed that the general force balance determining the motion of a spherical drop of arbitrary viscosity moving in a linear flow near a wall takes the form

$$\tfrac{4}{3}\pi\bar{\rho} Re\, St \frac{\mathrm{d}\boldsymbol{V}_B}{\mathrm{d}t} = \tfrac{4}{3}\pi\left[(\bar{\rho}-1)\boldsymbol{g} + Re\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t'}\right] - 4\pi R_\mu \boldsymbol{P}^{(1)}\cdot \boldsymbol{V}_{S0} + (2\boldsymbol{R}_V - \boldsymbol{R}_M):\boldsymbol{S} + \boldsymbol{F}_U + \boldsymbol{F}_L, \tag{26}$$

where $\boldsymbol{F}_U$ (resp. $\boldsymbol{F}_L$) represents the contribution of temporal acceleration (resp. the quasi-steady inertial contribution) of the flow disturbance to the total force, the $O(\kappa^3)$-approximation of $\boldsymbol{P}^{(1)}$, $\boldsymbol{R}_M$ and $\boldsymbol{R}_V$ being given by (A 3*a*), (A 7*a*) and (A 7*b*), respectively. The third term on the right-hand side is a Faxén correction resulting from the interaction of the undisturbed shear with the unsymmetrical distribution of the interfacial velocity and stress induced by the wall. The contribution of temporal acceleration in (26) is given by (14*b*), while the quasi-steady inertial contribution is given by (17) (resp. (21)) in a pure shear flow (resp. in a solid-body rotation flow). All three contributions have an $O(\kappa^{-1})$ leading-order term corresponding to an effect that already exists in an unbounded flow and is simply altered by the wall. The magnitude of this effect is reduced by the presence of the wall but its sign is left unchanged. For instance, the leading-order term of the unsteady contribution corresponds to the long-time history force in unbounded flow. The influence of the wall reduces the magnitude of this effect from $O((Re\, St)^{1/2})$ for $\kappa = O((Re\, St)^{1/2})$ to $O(Re\, St)$ for $\kappa = O(1)$ and the sign of each component of this force remains opposite to that of the corresponding component of the relative acceleration $\mathrm{d}\boldsymbol{V}_{S0}/\mathrm{d}t$. Similarly, in the presence of both a non-zero slip velocity and a non-zero shear (or rotation), there is an $O(\kappa^{-1})$ lift force with sign identical to that of the corresponding effect in an unbounded flow. In addition, the slip velocity and the shear induce $O(\kappa^0)$ forces normal to the wall that have no counterpart in an unbounded flow. By studying particular situations in which temporal acceleration and quasi-steady lift act in the same direction, we showed how the magnitude of the resulting lift force may be affected by the former effects. This is particularly spectacular in the case of a rotating flow where, for a given rotation rate and slip velocity, the lift force experienced by a drop whose motion is driven by the centrifugal force is about six times larger than that on the same drop in motion under the effect of a body force with a constant direction.

Throughout this investigation the viscosity and density ratios λ and $\bar{\rho}$ had arbitrary values. The influence of λ on the leading-order results (terms of $O(\kappa^{-1})$) is always manifested in a factor R_μ^2 where R_μ is the strength of the Stokeslet involved in the unbounded solution. The influence of λ on $O(\kappa^0)$ terms is more complicated but does not change the order of magnitude of the corresponding contributions. The density ratio $\bar{\rho}$ was found to influence the force $\boldsymbol{F}_U$ due to temporal acceleration through a term proportional to $(1+\lambda)^{-2}$. Hence this influence vanishes for highly viscous drops and rigid particles. Moreover, owing to symmetry, $\bar{\rho}$ does not appear in (17) and (21), which means that the drop density does not affect the quasi-steady lift force at the present order of approximation. Table 1 summarizes the expressions for the various contributions to the hydrodynamic force in the two limit cases of an inviscid massless bubble and a rigid sphere. Provided conditions (6) are satisfied, the results in this table for the contribution of temporal acceleration and those for slip and shear/rotation may be added to obtain a force balance correct up to order min($O(Re)$, $O(Re\,St)$).

Throughout this work, results were obtained in a non-inertial frame of reference translating with the drop and it is important to determine how they transform in the inertial laboratory frame. First, (26) remains unchanged under such a transformation because $\mathrm{d}\boldsymbol{V}_B/\mathrm{d}t = \mathrm{d}\boldsymbol{V}_B/\mathrm{d}t'$ (the prime denoting differentiation in the inertial frame). Similarly, (17) and (21) are unchanged since they just involve the slip velocity $\boldsymbol{V}_{S0}$ and velocity gradients. In contrast, the acceleration $\mathrm{d}\boldsymbol{V}_{S0}/\mathrm{d}t$ becomes $\mathrm{d}\boldsymbol{V}_{S0}/\mathrm{d}t = \mathrm{d}\boldsymbol{V}_B/\mathrm{d}t' - (\mathrm{d}\boldsymbol{V}/\mathrm{d}t' + \boldsymbol{V}_B \cdot \nabla'\boldsymbol{V})$ (see (5)). Hence we can conclude that the long-time history force present in (14a, b) (i.e. the $O(\kappa^{-1})$ term) involves the difference between the time variation of the fluid velocity 'seen' by the particle, $\mathrm{d}\boldsymbol{V}/\mathrm{d}t' + \boldsymbol{V}_B \cdot \nabla'\boldsymbol{V}$, and the particle acceleration, $\mathrm{d}\boldsymbol{V}_B/\mathrm{d}t'$. This conclusion is consistent with that of Magnaudet, Rivero & Fabre (1995) concerning the history force in inhomogeneous flows.

The present results are influenced by the long-time added-mass force at $O(\kappa^0)$. However, this effect provides only a second-order contribution in the present context, so that there is no flow configuration in which added-mass acts alone. In (14a, b) it combines with the long-time history force (and both effects are proportional to the same 'instantaneous' acceleration because we only considered time variations operating on $O((Re\,St)^{-1})$ time scales), while in (17) and (21) added-mass due to convective acceleration combines with the shear-induced (or rotation-induced) lift force. This impossibility of isolating added-mass effects contrasts with the situation encountered in unsteady Stokes flow (where the added-mass contribution is the only one involving the instantaneous acceleration) or in irrotational inviscid flow where the whole hydrodynamic force (except the contribution of buoyancy) is due to added-mass. This does not affect the results derived above but its conceptual consequence is that the value of the added-mass coefficient cannot be determined *per se* in the present context. Because of this, it is not possible to specify the general expression for the acceleration involved in the added-mass force, unlike the inviscid limit where it is known that this force is proportional to $\mathrm{D}\boldsymbol{V}/\mathrm{D}t' - \mathrm{d}\boldsymbol{V}_B/\mathrm{d}t'$ for a spherical body (Taylor 1928; Auton, Hunt & Prud'homme 1988). More precisely, as we saw before, terms of $O(\kappa^0)$ in (14a) are proportional to $-St\,\mathrm{d}\boldsymbol{V}_{S0}/\mathrm{d}t$, i.e. to $(St\,\mathrm{d}\boldsymbol{V}/\mathrm{d}t' + \boldsymbol{V}_B \cdot \nabla'\boldsymbol{V}) - St\,\mathrm{d}\boldsymbol{V}_B/\mathrm{d}t'$ when expressed in an inertial frame of reference. Terms $-\alpha(\boldsymbol{V}_{S0} \cdot \boldsymbol{e}_3)\boldsymbol{e}_1$ in (17) and $-\Omega\boldsymbol{e}_3 \times \boldsymbol{V}_{S0}$ in (21) may also be re-expressed by introducing the velocity gradient $\nabla\boldsymbol{V}$; this allows us to write them in the generic form $-\boldsymbol{V}_{S0} \cdot \nabla\boldsymbol{V} = (\boldsymbol{V} - \boldsymbol{V}_B) \cdot \nabla'\boldsymbol{V}$. Hence, on adding the above forms of $-St\,\mathrm{d}\boldsymbol{V}_{S0}/\mathrm{d}t$ and $-\alpha(\boldsymbol{V}_{S0} \cdot \boldsymbol{e}_3)\boldsymbol{e}_1$ or $-\Omega\boldsymbol{e}_3 \times \boldsymbol{V}_{S0}$ and using (5), we see that any inertial $O(\kappa^0)$ effect whose contributions to (14a) and to (17) or (21) involve the same numerical prefactor is actually described by a single expression proportional to the relative acceleration $\mathrm{D}\boldsymbol{V}/\mathrm{D}t' - St\,\mathrm{d}\boldsymbol{V}_B/\mathrm{d}t'$. If we were able to demonstrate that the contributions of added-mass to $\boldsymbol{F}_U$ and $\boldsymbol{F}_L$ involve the same coefficient, we could then conclude that the low-Re expression for the added-mass force is proportional to the difference between the local acceleration of the undisturbed flow and that of the drop, as it is in the inviscid limit. However, since we cannot identify unambiguously the added-mass force because we assumed $Re\,St \ll 1$, we cannot evaluate its contributions to $\boldsymbol{F}_U$ and $\boldsymbol{F}_L$, and the above representation remains a pure conjecture at this stage, even though there is *a priori* no reason to believe that the wall changes the value of the added-mass coefficient at the present order of approximation (according to irrotational flow theory, added-mass effects are affected by a wall only at $O(\kappa^3)$, see Milne-Thomson 1968, p. 563).

An interesting feature of the present results is that the quasi-steady inertial contribution $\boldsymbol{F}_L$ always reduces to a sideways (or lift) force. In other words, unlike the well-known Oseen correction, we did not find any quasi-steady inertial correction to the drag at the present order of approximation. This may seem surprising since it is well-known that $O((\alpha Re)^{1/2})$ or $O((Ta)^{1/2})$ drag corrections usually arise with the inertial lift force in linear flows (see Saffman 1965; Harper & Chang 1968; Herron *et al.* 1975, Gotoh 1990; Perez-Madrid *et al.* 1990). Such corrections also occur when the particle slips along a direction perpendicular to a plane shear flow (Harper & Chang 1968) or along the axis of a rotating flow (Childress 1964; Weiserborn 1985). The reason why such drag corrections do not occur here may be understood by considering first the unbounded situation. As pointed out by Lovalenti & Brady (1993*a*), the outer expansion generally contributes to a quasi-steady drag correction. In contrast, owing to geometrical symmetries in the volume integrals associated with the inner expansion, the regular part of the expansion can only contribute to a sideways force. As we showed in § 3, because of the presence of the wall the leading-order contribution to the hydrodynamic force does not come from the outer expansion in the situations considered here. Moroever, it can be checked that the integrands involved in the volume integrals of (8) are always either odd functions of x_1 and/or x_2 or functions that integrate to zero between $x_3 = -1/\kappa$ and $x_3 \to +\infty$; in both cases this prevents the occurrence of any non-zero quasi-steady drag correction at the present order of approximation. In contrast, such a correction would arise if we were considering second-order inertial effects because evaluation of these effects involves the outer expansion, as showed by Cox & Hsu (1977).

The present results are of direct use for evaluating inertial forces on a drop in low-Reynolds-number motion near a wall, provided conditions $\kappa \ll 1$, $Re\,St \ll 1$ (see § 2), $Re \ll St \ll Re^{-1/2}$ and $\kappa^{-1} < \min(l_u, l_s, l_\alpha, l_\Omega)$ (see § 3) are satisfied. These results may for instance be incorporated in a Lagrangian procedure to track bubbles, drops or rigid particles moving near a wall. However, owing to the last of the above conditions, they cannot match directly results valid in unbounded flow, like those of Lovalenti & Brady (1993*a*, *b*), Harper & Chang (1968), Herron *et al.* (1975) or Gotoh (1990). More precisely, the scalings of both series of expressions match when the wall lies near the outer limit of the Stokes region (as we saw in §§ 5–7), but there is no reason for the numerical prefactors involved in the two series of expressions to be identical. To obtain such a quantitative matching on rational grounds, it would be necessary to reconsider the effects discussed in the present work in the intermediate situation $\kappa^{-1} > \min(l_u, l_s, l_\alpha, l_\Omega)$ where the wall lies in the outer region of the flow disturbance. Then the leading-order effect of the wall would come from the outer expansion of the disturbance, and its contribution to the drag and lift components of the hydrodynamic force would appear in the form of integrals depending on the dimensionless distances $(\kappa l_u)^{-1}$, $(\kappa l_s)^{-1}$, $(\kappa l_\alpha)^{-1}$ or $(\kappa l_\Omega)^{-1}$. To our knowledge, results in this regime are available only in the two cases of the slip-induced lift and drag forces in a quiescent fluid (Vasseur & Cox 1977), and of the shear-induced lift force for the particular situation where the leading-order slip velocity lies along the streamlines of the undisturbed flow (McLaughlin 1993). These results were derived for a rigid sphere, but the analysis of Legendre & Magnaudet (1997) and the direct calculation presented by Takemura *et al.* (2002) for an inviscid bubble moving parallel to a wall in a quiescent fluid show that they can be readily extended to a drop of arbitrary viscosity by multiplying the strength of the inertial correction to the force by a factor R_μ^2. It would be very desirable that results similar to those of Vasseur & Cox (1977) and McLaughlin (1993) be derived for the contribution of temporal acceleration as well as for the case of a particle translating in an arbitrary direction with respect to a shear flow or a solid-body rotation flow. Note that this leads to challenging problems, since for instance determining the contribution of temporal acceleration for the case where the wall lies in the outer region requires the solution of the unsteady counterpart of the problem considered by Vasseur & Cox (1977). If such results are made available, they could be combined with those already known for unbounded flows and those derived in the present investigation for the near-wall region. This would allow us to obtain a unified description of low-Reynolds-number inertial effects experienced by a drop or a particle in both unbounded and wall-bounded linear flows, and such a description would have an important bearing on the prediction of many two-phase flows of practical interest.

Appendix A. Solution of the auxiliary problem

Near the drop, i.e. for $r \ll \kappa^{-1}$, the $O(\kappa^3)$-solution of (7) may be expressed in the compact form

$$\left.\begin{aligned} \widehat{\boldsymbol{U}} &= \boldsymbol{e}\cdot\left[\boldsymbol{P}^{(1)}\cdot\boldsymbol{M}^{(1)}+\boldsymbol{P}^{(2)}\cdot\boldsymbol{M}^{(2)}+\boldsymbol{P}^{(3)}\cdot\boldsymbol{M}^{(3)}\right]+O(\kappa^4),\\ \widehat{\tilde{\boldsymbol{U}}} &= \boldsymbol{e}\cdot\left[\boldsymbol{P}^{(1)}\cdot\tilde{\boldsymbol{M}}^{(1)}+\boldsymbol{P}^{(2)}\cdot\tilde{\boldsymbol{M}}^{(2)}+\boldsymbol{P}^{(3)}\cdot\tilde{\boldsymbol{M}}^{(3)}\right]+O(\kappa^4),\end{aligned}\right\} \tag{A 1}$$

where the $\boldsymbol{M}^{(i)}$ and $\tilde{\boldsymbol{M}}^{(i)}$ are second-rank tensors corresponding to the first three series of contributions in the multipole expansion of the disturbance flow induced by the drop, and the $\boldsymbol{P}^{(i)}$ are diagonal projectors accounting for the anisotropy of space introduced by the presence of the wall. We shall not explicitly write $\boldsymbol{M}^{(3)}$ and $\tilde{\boldsymbol{M}}^{(3)}$ because they do not contribute to the hydrodynamic force on the drop. Using the results of MTL, the expressions for the other quantities involved in (A 1) are found to be

$$\boldsymbol{M}^{(1)} = -\boldsymbol{I} + \frac{R_\mu}{2}\left(\frac{\boldsymbol{I}}{r}+\frac{\boldsymbol{xx}}{r^3}\right)+D_\mu\left(\frac{\boldsymbol{I}}{r^3}-3\frac{\boldsymbol{xx}}{r^5}\right), \tag{A 2a}$$

$$\tilde{\boldsymbol{M}}^{(1)} = M_\mu[(1-2r^2)\boldsymbol{I}+\boldsymbol{xx}], \tag{A 2b}$$

$$\begin{aligned}\boldsymbol{M}^{(2)} = {}& \tfrac{1}{3}\boldsymbol{e}_3\boldsymbol{x}-\tfrac{1}{2}(\boldsymbol{xe}_3+x_3\boldsymbol{I})+R_S\left(x_3\boldsymbol{x}-\tfrac{1}{3}r^2\boldsymbol{e}_3\right)\frac{\boldsymbol{x}}{r^5}\\ &+2D_\mu\left(\frac{\boldsymbol{xe}_3+\boldsymbol{e}_3\boldsymbol{x}+x_3\boldsymbol{I}}{r^5}-5\frac{x_3\boldsymbol{xx}}{r^7}\right),\end{aligned} \tag{A 2c}$$

$$\tilde{\boldsymbol{M}}^{(2)} = \frac{M_\mu}{2}[(3-5r^2)(x_3\boldsymbol{I}+\boldsymbol{xe}_3)+2(r^2-1)\boldsymbol{e}_3\boldsymbol{x}+4x_3\boldsymbol{xx}], \tag{A 2d}$$

$$\boldsymbol{P}^{(1)} = \left(\sum_{n=0}^{3}(-1)^n I_D^n - D_\mu I_0\right)(\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2) + \left(\sum_{n=0}^{3}(-1)^n J_D^n - D_\mu J_0\right)\boldsymbol{e}_3\boldsymbol{e}_3, \quad \text{(A 3}a\text{)}$$

$$\boldsymbol{P}^{(2)} = -2I_S(1 - I_D)(\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2) + 3J_S(1 - J_D)\boldsymbol{e}_3\boldsymbol{e}_3, \quad \text{(A 3}b\text{)}$$

with

$$R_\mu = \frac{2+3\lambda}{2(1+\lambda)}, \quad D_\mu = \frac{\lambda}{4(1+\lambda)}, \quad M_\mu = \frac{1}{2(1+\lambda)}, \quad R_S = \frac{2+5\lambda}{2(1+\lambda)}, \quad \text{(A 4)}$$

and

$$I_D = -\tfrac{3}{8}R_\mu\kappa, \quad I_S = \tfrac{3}{16}R_\mu\kappa^2, \quad J_D = -\tfrac{3}{4}R_\mu\kappa, \quad J_S = -\tfrac{3}{16}R_\mu\kappa^2, \quad I_0 = \tfrac{1}{2}\kappa^3, \quad J_0 = 2\kappa^3. \quad \text{(A 5)}$$

The coefficients defined in (A 4) depend on the viscosity ratio λ and characterize the strength of the various singularities involved in the expansion (A 1). Equations (A 5) express the influence of the wall on the flow disturbance in the vicinity of the drop. Obviously $\boldsymbol{P}^{(1)} \to \boldsymbol{I}$ and $\boldsymbol{P}^{(2)} \to 0$ as the separation between the drop and the wall tends to infinity.

From (A 2a) and (A 3a) we immediately deduce that the force experienced by the drop when it translates with a unit velocity in the direction defined by the unit vector $\boldsymbol{e}$ is

$$\int_{A_B} \hat{\boldsymbol{\Sigma}} \cdot \boldsymbol{e}_r \,\mathrm{d}S = \hat{\boldsymbol{F}}_D = -4\pi R_\mu e \cdot \boldsymbol{P}^{(1)} + O(\kappa^4) \quad \text{(A 6)}$$

so that the resistance tensor is just $4\pi R_\mu \boldsymbol{P}^{(1)}$. Note that since $I_D \neq J_D$ and $I_0 \neq J_0$, $\hat{\boldsymbol{F}}_D$ is generally not antiparallel to $\boldsymbol{e}$. In view of the evaluation of the Faxén force revealed by the reciprocal theorem, we also need to evaluate the moment $\int_{A_B} \boldsymbol{x}\hat{\boldsymbol{\Sigma}} \cdot \boldsymbol{e}_r \,\mathrm{d}S$. Introducing the third-rank tensor $\boldsymbol{R}_M$ such that $\int_{A_B} \boldsymbol{x}\hat{\boldsymbol{\Sigma}} \cdot \boldsymbol{e}_r \,\mathrm{d}S = \boldsymbol{e} \cdot \boldsymbol{R}_M$, we obtain for the geometry considered here, i.e. a wall normal to the x_3-direction,

$$\boldsymbol{R}_M = \frac{4}{5}\pi\left\{\frac{16+25\lambda}{3(1+\lambda)} I_S(1 - I_D)\boldsymbol{e}_1(\boldsymbol{e}_1\boldsymbol{e}_3 + \boldsymbol{e}_3\boldsymbol{e}_1) + J_S(1 - J_D)\boldsymbol{e}_3\left[\frac{2}{1+\lambda}(\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2) - \frac{14+25\lambda}{1+\lambda}\boldsymbol{e}_3\boldsymbol{e}_3\right]\right\} + O(\kappa^4). \quad \text{(A 7}a\text{)}$$

For the same reason we also need the quantity $\int_{A_B} \hat{\boldsymbol{U}}\boldsymbol{e}_r \,\mathrm{d}S$, which by virtue of the symmetry properties of $\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(3)}$ equals $\boldsymbol{e} \cdot \boldsymbol{P}^{(2)} \cdot \int_{A_B} \boldsymbol{M}^{(2)}\boldsymbol{e}_r \,\mathrm{d}S + O(\kappa^4)$. Again we define a third-rank tensor $\boldsymbol{R}_V$ such that $\boldsymbol{R}_V = \boldsymbol{P}^{(2)} \cdot \int_{A_B} \boldsymbol{M}^{(2)}\boldsymbol{e}_r \,\mathrm{d}S$. After some algebra we find

$$\boldsymbol{R}_V = \frac{4}{5(1+\lambda)}\pi\left\{I_S(1 - I_D)\boldsymbol{e}_1(\boldsymbol{e}_1\boldsymbol{e}_3 + \boldsymbol{e}_3\boldsymbol{e}_1) + J_S(1 - J_D)\boldsymbol{e}_3[(\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2) - 2\boldsymbol{e}_3\boldsymbol{e}_3]\right\}. \quad \text{(A 7}b\text{)}$$

The solution (A 1)–(A 5) for $\hat{\boldsymbol{U}}$ is only valid near the drop. To obtain a uniformly valid solution we introduce the strained coordinates $\overline{x_i} = \kappa x_i$ $(i = 1, 3)$, $\overline{r} = \kappa r$ (note that in this coordinate system the wall is located at $\overline{x_3} = -1$). Using again the results of MTL we may write the general $O(\kappa^2)$-expression for the auxiliary velocity field $\hat{\boldsymbol{U}}$ in the form

$$\overline{\hat{\boldsymbol{U}}} + \boldsymbol{e} = \boldsymbol{e} \cdot \left(\boldsymbol{P}_1^{(1)} \cdot \overline{\boldsymbol{M}}_1^{(1)} + \boldsymbol{P}_2^{(1)} \cdot \overline{\boldsymbol{M}}_2^{(1)}\right) + O(\kappa^3), \quad \text{(A 8)}$$

with

$$\overline{\mathbf{M}}_1^{(1)} = \frac{\kappa}{2} R_\mu \left\{ \left(\frac{\mathbf{I}}{\overline{r}} + \frac{\overline{\boldsymbol{x}}\,\overline{\boldsymbol{x}}}{\overline{r}^3} \right) - \left(\frac{\mathbf{I}}{\overline{\tau}} + \frac{\overline{\boldsymbol{X}}\,\overline{\boldsymbol{X}}}{\overline{\tau}^3} \right) \right\}, \tag{A 9a}$$

$$\overline{\mathbf{M}}_2^{(1)} = \kappa R_\mu \left[(1 + \overline{x_3}) \left(\frac{\mathbf{I}}{\overline{\tau}^3} - \frac{\overline{\boldsymbol{X}}\,\overline{\boldsymbol{X}}}{\overline{\tau}^5} \right) + \frac{\boldsymbol{e}_3 \overline{\boldsymbol{X}} - \overline{\boldsymbol{X}} \boldsymbol{e}_3}{\overline{\tau}^3} \right], \tag{A 9b}$$

$$\mathbf{P}_1^{(1)} = (1 - I_D)(\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2) + (1 - J_D)\boldsymbol{e}_3\boldsymbol{e}_3, \tag{A 10a}$$

$$\mathbf{P}_2^{(1)} = (1 - I_D)(\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2) - (1 - J_D)\boldsymbol{e}_3\boldsymbol{e}_3, \tag{A 10b}$$

with $\overline{\tau} = (\overline{x_1}^2 + \overline{x_2}^2 + (\overline{x_3} + 2)^2)^{1/2}$, $\overline{\boldsymbol{x}} = \overline{x_1}\boldsymbol{e}_1 + \overline{x_2}\boldsymbol{e}_2 + \overline{x_3}\boldsymbol{e}_3$ and $\overline{\boldsymbol{X}} = \overline{\boldsymbol{x}} + 2$. The velocity field resulting from (A 9*a*) is the difference between the disturbance due to the drop and that due to its image located at $\overline{x_1} = \overline{x_2} = 0, \overline{x_3} = -2$. The velocity associated with (A 9*b*) is an additional field required to satisfy the no-slip boundary condition at the wall.

Appendix B. Derivation of the reciprocal theorem

Most of the intermediate steps required in the derivation of the reciprocal theorem are similar to those detailed by Lovalenti & Brady (1993*a*, *b*). Consequently, only the main steps and the specific aspects due to the presence of the wall are indicated here.

We start from the momentum balances outside and inside the drop as given by (1), and combine them with the momentum equations of the auxiliary problem (7). Using the divergence theorem and integrating over the corresponding fluid domain yields

$$\int_{A_B \cup A_\infty \cup A_W} [(\widehat{\boldsymbol{U}} + \boldsymbol{e}) \cdot \boldsymbol{\Sigma} - (\boldsymbol{U} + \boldsymbol{V}_B) \cdot \hat{\boldsymbol{\Sigma}}] \cdot \boldsymbol{n}_e \,\mathrm{d}S = Re \int_{V_F} (\widehat{\boldsymbol{U}} + \boldsymbol{e}) \cdot \left(St \frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} \right) \mathrm{d}V, \tag{B 1a}$$

$$\lambda \int_{A_B} (\hat{\tilde{\boldsymbol{U}}} \cdot \tilde{\boldsymbol{\Sigma}} - \tilde{\boldsymbol{U}} \cdot \hat{\tilde{\boldsymbol{\Sigma}}}) \cdot \boldsymbol{n} \,\mathrm{d}S = \bar{\rho} Re \int_{V_B} \hat{\tilde{\boldsymbol{U}}} \cdot \left(St \frac{\partial \tilde{\boldsymbol{U}}}{\partial t} + (\tilde{\boldsymbol{U}} \cdot \nabla)\, \tilde{\boldsymbol{U}} \right) \mathrm{d}V, \tag{B 1b}$$

where A_W and A_∞ denote the wall and the outer boundary of the fluid domain $\boldsymbol{V}_F$ surrounding the drop, respectively, and $\boldsymbol{n}_e$ is the unit normal to V_F directed outward. Then, adding (B 1*a*) and (B 1*b*), using the boundary conditions at the drop surface (see (1)) and the force balance (2), we obtain

$$\begin{aligned} \hat{\boldsymbol{F}}_D \cdot \boldsymbol{V}_B + \frac{4}{3}\pi(\bar{\rho} - 1)\boldsymbol{e} \cdot \boldsymbol{F} + \int_{A_\infty \cup A_W} [(\widehat{\boldsymbol{U}} + \boldsymbol{e}) \cdot \boldsymbol{\Sigma} - (\boldsymbol{U} + \boldsymbol{V}_B) \cdot \hat{\boldsymbol{\Sigma}}] \cdot \boldsymbol{n}_e \,\mathrm{d}S \\ = Re \int_{V_F} (\widehat{\boldsymbol{U}} + \boldsymbol{e}) \cdot \left(St \frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} \right) \mathrm{d}V \\ + \bar{\rho} Re \int_{V_B} \hat{\tilde{\boldsymbol{U}}} \cdot \left(St \frac{\partial \tilde{\boldsymbol{U}}}{\partial t} + (\tilde{\boldsymbol{U}} \cdot \nabla)\tilde{\boldsymbol{U}} \right) \mathrm{d}V, \end{aligned} \tag{B 2}$$

with $\hat{\boldsymbol{F}}_D = \int_{A_B} \hat{\boldsymbol{\Sigma}} \cdot \boldsymbol{e}_r \,\mathrm{d}S$. We now use the decompositions $\boldsymbol{U} = \boldsymbol{V} - \boldsymbol{V}_B + \boldsymbol{u}$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 + \boldsymbol{\sigma}$ introduced in § 3 and require that $\|\boldsymbol{u}\| = O(r^{-\beta})$ and $\|\boldsymbol{\sigma}\| = O(r^{-\beta-1})$ with $\beta > 0$ for $r \to \infty$, which is obviously satisfied. Then, after several manipulations and uses of the divergence theorem, the integral over $A_W \cup A_\infty$ in (B 2) may be written in the form

$$\int_{A_\infty \cup A_W} [(\widehat{\boldsymbol{U}} + \boldsymbol{e}) \cdot \boldsymbol{\Sigma} - (\boldsymbol{U} + \boldsymbol{V}_B) \cdot \hat{\boldsymbol{\Sigma}}] \cdot \boldsymbol{n}_e \,\mathrm{d}S = -\hat{\boldsymbol{F}}_D \cdot \boldsymbol{V}_0$$
$$+ \int_{A_B} [(\widehat{\boldsymbol{U}} + \boldsymbol{e}) \cdot \boldsymbol{\Sigma}_0 - (\boldsymbol{x} \cdot \nabla)\boldsymbol{V} \cdot \hat{\boldsymbol{\Sigma}}] \cdot \boldsymbol{n} \,\mathrm{d}S$$
$$+ Re \int_{V_F} (\widehat{\boldsymbol{U}} + \boldsymbol{e}) \cdot \left(\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t'} - St \frac{\mathrm{d}\boldsymbol{V}_B}{\mathrm{d}t} \right) \mathrm{d}V, \qquad \text{(B 3)}$$

where $\boldsymbol{V}_0$ is the value of the undisturbed velocity $\boldsymbol{V}$ at $\boldsymbol{x} = 0$ and we have assumed that the undisturbed flow depends linearly on the local position, as it does in flows defined by (3) and (4). Combining (5) and (B 3) in (B 2), using the kinematic condition at the drop surface, the definition of $\boldsymbol{F}$ given in § 2, and noting that in the auxiliary problem the net torque on the drop is zero, we obtain

$$\frac{4}{3}\pi \bar{\rho} Re\, St\, \boldsymbol{e} \cdot \frac{\mathrm{d}\boldsymbol{V}_B}{\mathrm{d}t} = \frac{4}{3}\pi \boldsymbol{e} \cdot \left[(\bar{\rho} - 1)\boldsymbol{g} + Re \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t'} \right]$$
$$+ \hat{\boldsymbol{F}}_D \cdot \boldsymbol{V}_{S0} + \mathbf{S} : \int_{A_B} [2\widehat{\boldsymbol{U}}\boldsymbol{n} - \boldsymbol{x}\hat{\boldsymbol{\Sigma}} \cdot \boldsymbol{n}] \,\mathrm{d}S - Re \int_{V_F} (\widehat{\boldsymbol{U}} + \boldsymbol{e})$$
$$\cdot \left(St \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{V} \right) \mathrm{d}V - \bar{\rho} Re \int_{V_B} \hat{\tilde{\boldsymbol{U}}}$$
$$\cdot \left(St \frac{\partial \tilde{\boldsymbol{U}}}{\partial t} + (\tilde{\boldsymbol{U}} \cdot \nabla)\tilde{\boldsymbol{U}} \right) \mathrm{d}V, \qquad \text{(B 4)}$$

where we have defined the slip velocity at the centre of the drop as $\boldsymbol{V}_{S0} = \boldsymbol{V}_B - \boldsymbol{V}_0$.

Appendix C. The velocity disturbances

Here we use the results of MTL to specify the form of the disturbances $\boldsymbol{u}$ and $\tilde{\boldsymbol{U}}$ corresponding to the undisturbed flow fields defined by (3) and (4).

For the first of these (linear shear), the $O(\kappa^0)$-approximation near the drop is

$$\boldsymbol{u}_\alpha = \boldsymbol{V}_{S\alpha}(t) \cdot \left(\mathbf{I} + \mathbf{M}^{(1)} \right) - \alpha \left(R_S \frac{x_1 x_3 \boldsymbol{x}}{r^5} + 2D_\mu \left(\frac{x_1 \boldsymbol{e}_3 + x_3 \boldsymbol{e}_1}{r^5} - 5\frac{x_1 x_3 \boldsymbol{x}}{r^7} \right) \right), \qquad \text{(C 1}a\text{)}$$

$$\tilde{\boldsymbol{U}}_\alpha = \boldsymbol{V}_{S\alpha}(t) \cdot \tilde{\mathbf{M}}^{(1)} - \alpha \frac{M_\mu}{2} [(3 - 5r^2)(x_3 \boldsymbol{e}_1 + x_1 \boldsymbol{e}_3) + 4x_1 x_3 \boldsymbol{x}], \qquad \text{(C 1}b\text{)}$$

where the slip velocity is $\boldsymbol{V}_{S\alpha}(t) = \boldsymbol{V}_W(t) - \boldsymbol{V}_B(t) + (\alpha/\kappa(t))\boldsymbol{e}_1$ and definitions (A 2) are used.

Similarly, in the case of a solid-body rotation about the x_3-direction we have

$$\boldsymbol{u}_\Omega = \boldsymbol{V}_{S\Omega}(t) \cdot \left(\mathbf{I} + \mathbf{M}^{(1)} \right), \qquad \text{(C 2}a\text{)}$$

$$\tilde{\boldsymbol{U}}_\Omega = \boldsymbol{V}_{S\Omega}(t) \cdot \tilde{\mathbf{M}}^{(1)} + \Omega(x_1 \boldsymbol{e}_2 - x_2 \boldsymbol{e}_1), \qquad \text{(C 2}b\text{)}$$

with $\boldsymbol{V}_{S\Omega}(t) = \boldsymbol{V}_W(t) - \boldsymbol{V}_B(t)$, $\boldsymbol{V}_W(t)$ being defined by (4).

In strained coordinates, the uniformly valid $O(\kappa^2)$-expression $\bar{\boldsymbol{u}}_\alpha$ (resp. $\bar{\boldsymbol{u}}_\Omega$) for $\boldsymbol{u}_\alpha$ (resp. $\boldsymbol{u}_\Omega$) is

$$\bar{\boldsymbol{u}}_\alpha = \boldsymbol{V}_{S\alpha}(t) \cdot \left(\mathbf{P}_{\kappa 1} \cdot \overline{\mathbf{M}}_1^{(1)} + \mathbf{P}_{\kappa 2} \cdot \overline{\mathbf{M}}_2^{(1)} \right) - 2\alpha R_S \bar{\boldsymbol{u}}_{Str} + O(\kappa^3), \qquad \text{(C 3}a\text{)}$$

$$\bar{\boldsymbol{u}}_\Omega = \boldsymbol{V}_{S\Omega}(t) \cdot \left(\mathbf{P}_{\kappa 1} \cdot \overline{\mathbf{M}}_1^{(1)} + \mathbf{P}_{\kappa 2} \cdot \overline{\mathbf{M}}_2^{(1)} \right) + O(\kappa^3), \qquad \text{(C 3}b\text{)}$$

where

$$\bar{\boldsymbol{u}}_{Str} = \kappa^2 \left\{ \left(\frac{\overline{x_3}}{\overline{r}^5} + \frac{2+\overline{x_3}}{\overline{\tau}^5} \right) \overline{x_1}\, \overline{\boldsymbol{x}} + 2\frac{1+\overline{x_3}}{\overline{\tau}^7} [\overline{\tau}^2((2+\overline{x_3})\boldsymbol{e}_1 + \overline{x_1}\boldsymbol{e}_3) - 5\overline{x_1}(2+\overline{x_3})(\overline{\boldsymbol{x}} + 2\boldsymbol{e}_3)] \right\} \quad \text{(C 4)}$$

and definitions (A 9) are used.

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